General Relativity: Exercises

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Listed below are the exercises that have been assigned during the course and collected according to the lecture in which they were assigned. These exercises can be solved independently or together during the exercise time. Some of these questions could be part of the oral exam.

Lecture I

1. Consider a binary systems of gravitating objects of mass M and m.

    - Consider first the case in which $m \ll M$ and where the small-mass object is in quasi-circular orbit around the more massive one. Draw the trajectory in two-space and the worldline in a $1+1$- and in a $2+1$-dimensional spacetime [Hint: use a coordinate system centered in M].
    - Let now $m = M$ and the binary orbit in circular orbit around the Newtonian center of mass of the system. Draw the trajectory in two-space and the worldline in a $1+1$- and in a $2+1$-dimensional spacetime [Hint: use a coordinate system centered in the Newtonian center of mass].

2. Consider a two-dimensional space and cover it with two coordinate maps: a Cartesian one $x^i = (x, y)$ and a polar one $x^{i'} = (r, \theta)$.

    - Find the coordinate transformation $\boldsymbol{f}: \ x^i \to x^{i'}$
    - Find the inverse coordinate transformation $\boldsymbol{f}^{-1}: \ x^{i'} \to x^i$
    - Find the components of the transformation matrix $\Lambda^{i'}_{\ i}$ and its determinant $J' := |\partial x^{i'}/\partial x^i|$.
    - Find the components of the inverse transformation matrix $\Lambda^{i}_{\ i'}$ and its determinant $J := |\partial x^i/\partial x^{i'}|$.
    - Show that $\Lambda^{i'}_{\ i}\,\Lambda^{i}_{\ j'} = \delta^{i'}_{\ \ j'}$ and that $J\,J' = 1$.

3. Consider a three-dimensional space and cover it with two coordinate maps: a Cartesian one $x^i = (x, y, z)$ and a polar one $x^{i'} = (r, \theta, \phi)$. Address all the questions in point 2.

Lecture II

1. Consider two coordinate systems in a two-dimensional space $\{x^i\} = (x, y)$ and $\{x^{i'}\} = (r, \theta)$ that are related through the well-known coordinate transformation

$$\boldsymbol{f} : \begin{cases} r = \left(x^2 + y^2\right)^{1/2} \\ \theta = \tan^{-1}(y/x) \end{cases} \tag{1}$$

and its inverse

$$\boldsymbol{f} : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{2}$$

Discuss the differences between the transformation matrix employed to transform a covector in this space

$$\left(\tilde{dx}\right)_i = \Lambda^{i'}{}_i \left(\tilde{dx}\right)_{i'} , \tag{3}$$

and the one employed in the coordinate transformation

$$x^{i'} = \Lambda^{i'}{}_i \, x^i \, . \tag{4}$$

2. Consider two coordinate systems in a four-dimensional spacetime $\{x^\mu\} = (t, x, y, z)$ and $\{x^{\mu'}\} = (u, v, y, z)$ that are related through the coordinate transformation

$$\boldsymbol{f} : \begin{cases} u = t - x \\ v = t + x \end{cases} \tag{5}$$

and its inverse

$$\boldsymbol{f}^{-1} : \begin{cases} t = \frac{1}{2}(v + u) \\ \\ x = \frac{1}{2}(v - u) \end{cases} \tag{6}$$

   - Compute the matrices employed in the transformations

$$x^{\mu'} = \Lambda^{\mu'}{}_\mu \, x^\mu \qquad\qquad x^\mu = \Lambda^\mu{}_{\mu'} \, x^{\mu'} \, . \tag{7}$$

   - Consider a four-vector with components $U^\mu = (1, 0, 0, 0)$ in the coordinate system x^μ and compute the new components $U^{\mu'}$ in the coordinate system $x^{\mu'}$.
   - Repeat the calculation for the new vector $V^\mu = (-1/2, 1/2, 0, 0)$. Interpret the results.

3. Consider a $1 + 1$ representation of the sub-spaces with two coordinate systems (t, x) and (u, v).

   - Draw in the two spacetimes the worldline of a particle with velocity $\dot{x} := dx/dt = 0$.
   - Draw in the two spacetimes the worldline of a particle with velocity $\dot{x} = k$ $(x = kt)$ with $k < 1$.
   - Interpret the results.

Lecture III

1. Consider $\boldsymbol{T}$ as contravariant tensor of rank 2 with components $T^{\mu\nu}$. Under what conditions can this tensor be cast as the product of two contravariant vectors $\boldsymbol{U}$ and $\boldsymbol{V}$, *i.e.*, such that $T^{\mu\nu} = U^\mu \, V^\nu$?

2. Consider the following equation

$$T^{\mu\nu} = U^\mu + V^\nu \, . \tag{1}$$

   Is $\boldsymbol{T}$ a generic tensor?

3. Consider $\boldsymbol{F}$ as a tensor of rank 2 with covariant components $F_{\mu\nu}$ and that is antisymmetric in one coordinate system, *i.e.*, $F_{\mu\nu} = -F_{\nu\mu}$.

   - Show that $F_{\mu\nu}$ is antisymmetric in all coordinate systems.
   - Does the antisymmetry in the covariant indices apply also to the contravariant indices?
   - If so, show that $F^{\mu\nu}$ is antisymmetric in all coordinate systems.

4. Consider the antisymmetric tensor $A_{\mu\nu}$ such that $A_{\mu\nu} = -A_{\nu\mu}$ and the symmetric tensor $B^{\mu\nu}$ such that $B^{\mu\nu} = B^{\nu\mu}$. Prove the following identities:

$$A_{\mu\nu} B^{\mu\nu} = 0 \, , \tag{2}$$

$$V^{\mu\nu} A_{\mu\nu} = \frac{1}{2} \left(V^{\mu\nu} - V^{\nu\mu} \right) A_{\mu\nu} \, , \tag{3}$$

$$V^{\mu\nu} B_{\mu\nu} = \frac{1}{2} \left(V^{\mu\nu} + V^{\nu\mu} \right) B_{\mu\nu} \, , \tag{4}$$

   where $\boldsymbol{V}$ is a generic tensor of rank 2.

Lecture IV

1. Using a coordinate system (t, r, θ, ϕ), consider the metric line element given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right], \tag{1}$$

   where $\kappa = -1, 0, 1$.

   - Show that in a new coordinate system (t, χ, θ, ϕ) the line element (1) can be rewritten as

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + f^2(\chi)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]. \tag{2}$$

   - Find the form of the function $f(\chi)$ for $\kappa = -1, 0, 1$.
   - Discuss the properties of the metric in the case of $\kappa = 0$ [Hint: two metrics $\boldsymbol{g}$ and $\boldsymbol{g}'$ are conformally related if it is possible to express them as $\boldsymbol{g} = \Omega\, \boldsymbol{g}'$, where $\Omega = \Omega(x^\mu)$ is a generic function and is referred to as the *conformal factor*].

2. Using a coordinate system $(\eta, \chi, \theta, \phi)$, consider the metric line element given by

$$ds^2 = \Omega^2\left[-d\eta^2 + d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]. \tag{3}$$

   Consider now a new coordinate system $(\tau, \rho, \theta, \phi)$ where

$$\tau = \frac{2\sin\eta}{\cos\chi + \cos\eta}, \tag{4}$$

$$\rho = \frac{2\sin\chi}{\cos\chi + \cos\eta}, \tag{5}$$

   and find the expression of the metric (3) in the new coordinate system. Discuss the properties of the new metric.

3. Given the four-vector $\boldsymbol{u}$ such $u^\alpha u_\alpha = -1$ and the tensor $h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu$, prove the following identities

$$h_{\mu\nu}u^\mu = 0, \qquad h^\mu{}_\nu\, h^\lambda{}_\mu = h^\lambda{}_\nu, \qquad h^\mu{}_\mu = 3. \tag{6}$$

4. Consider the following antisymmetric tensor

$$F_{\alpha\beta} = -2E_{[\alpha}u_{\beta]} + \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma\, u_\delta. \tag{7}$$

   Express the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ in terms of the tensor $\boldsymbol{F}$ [Hint: contract $F_{\alpha\beta}$ with u^β and $\epsilon^{\alpha\beta\gamma\delta}$, respectively].

Lecture V

1. Let $\boldsymbol{F}$ be a rank-2 antisymmetric tensor, $\boldsymbol{G}$ is a rank-2 symmetric tensor, and $\boldsymbol{X}$ a rank-3 antisymmetric tensor. Provide explicit expressions for the following tensors: $F_{\mu\nu}$, $F_{[\mu\nu]}$, $F_{(\mu\nu)}$, $G_{[\mu\nu]}$, $G_{(\mu\nu)}$, $X_{[\alpha\beta\gamma]}$, $X_{(\alpha\beta\gamma)}$, $X_{[\alpha\beta]\gamma}$, $X_{(\alpha\beta)\gamma}$, $X_{[\alpha\beta](\gamma)}$, $X_{(\alpha\beta)[\gamma]}$.

2. Prove the following identities:

$$X_{((\alpha_1\ \alpha_2 \ldots \alpha_n))} = X_{(\alpha_1\ \alpha_2 \ldots \alpha_n)}\,, \tag{1}$$

$$X_{[[\alpha_1\ \alpha_2 \ldots \alpha_n]]} = X_{[\alpha_1\ \alpha_2 \ldots \alpha_n]}\,, \tag{2}$$

$$X_{(\alpha_1 \ldots [\alpha_l\ \alpha_m] \ldots \alpha_n)} = 0\,, \tag{3}$$

$$X_{[\alpha_1 \ldots [\alpha_l\ \alpha_m] \ldots \alpha_n]} = X_{[\alpha_1 \ldots \alpha_l\ \alpha_m \ldots \alpha_n]}\,, \tag{4}$$

3. Let $\boldsymbol{F}$ be a rank-2 antisymmetric tensor with components $F^{\mu\nu}$. From $\boldsymbol{F}$ construct another rank-2 antisymmetric tensor ${}^*\boldsymbol{F}$ such that

$${}^*\boldsymbol{F} := \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}\boldsymbol{e}_\mu \otimes \boldsymbol{e}_\nu\,. \tag{5}$$

The tensor ${}^*\boldsymbol{F}$ is normally referred to as the *dual* of ${}^*\boldsymbol{F}$. Show the following is true

$${}^*({}^*\boldsymbol{F}) = -\boldsymbol{F}\,. \tag{6}$$

4. Let $\boldsymbol{V}$ be a rank-3 tensor with components $V^{\alpha\beta\gamma}$ and define

$$({}^*V)^{\alpha\beta\gamma} := V_\mu \epsilon^{\mu\alpha\beta\gamma}\,. \tag{7}$$

Show the following is true

$$V^\mu V_\mu = -\frac{1}{3!}({}^*V)^{\alpha\beta\gamma}\,({}^*V)_{\alpha\beta\gamma}\,. \tag{8}$$

Lecture VI

1. Define the skew tensor $\boldsymbol{F}$ as $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ and use the results of exercise 1. of the previous lecture to show that

$$F_{\mu\nu} := 2\,\partial_{[\mu} A_{\nu]}\,. \tag{1}$$

   Show that such definition implies that

$$F_{\alpha\beta,\nu} + F_{\beta\nu,\alpha} + F_{\nu\alpha,\beta} = 0\,. \tag{2}$$

2. Consider a vector $\boldsymbol{V}$ with components V^μ relative to a coordinate basis, *i.e.*,

$$\boldsymbol{V} = V^\mu \partial_\mu = V^\mu \boldsymbol{e}_\mu\,. \tag{3}$$

   Define an object given by the partial derivative of the components of $\boldsymbol{V}$, *i.e.*,

$$U_\nu{}^\mu := \partial_\nu V^\mu\,. \tag{4}$$

   Show that $U_\nu{}^\mu$ is not a tensor. What are the implications of this result? What can be done to construct a tensor out measuring the derivative of a tensor?

3. Consider a line element in three-dimensional space

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\,, \tag{5}$$

   with a coordinate basis $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi\}$.

   - Construct the corresponding orthonormal basis $\{\boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\}$
   - Compute the structure coefficients $C^\theta_{\hat{r}\hat{\theta}}$ and $C^\theta_{r\theta}$. What is the difference between the two?
   - Compute the structure coefficients $C^\theta_{\hat{r}\hat{\phi}}$, $C^\phi_{\hat{r}\hat{\phi}}$, $C^\theta_{\hat{\theta}\hat{\phi}}$, and $C^\phi_{\hat{\theta}\hat{\phi}}$. Are there other nonzero structure coefficients?

Lecture VII

1. Show that if $\boldsymbol{g}$ is the metric tensor, its covariant derivative is zero, *i.e.*,

$$\nabla_\lambda g_{\mu\nu} = 0\,. \tag{1}$$

2. Using the result of exercise 1., derive the following definition of the Christoffel symbols

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}\right)\,. \tag{2}$$

3. Prove the following identities:

$$\partial_\gamma g_{\alpha\beta} = \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma}\,, \tag{3}$$

$$g_{\alpha\mu}\partial_\gamma g^{\mu\beta} = -g^{\mu\beta}\partial_\gamma g_{\alpha\mu}\,, \tag{4}$$

$$\partial_\gamma g^{\alpha\beta} = -\Gamma^\alpha_{\mu\gamma} g^{\mu\beta} - \Gamma^\beta_{\mu\gamma} g^{\mu\alpha}\,, \tag{5}$$

$$\left(\ln|g|\right)_{,\alpha} = g^{\mu\nu} g_{\mu\nu,\alpha}\,, \tag{6}$$

$$\nabla_\mu A^\mu = \frac{1}{|g|^{1/2}} \partial_\mu \left(|g|^{1/2} A^\mu\right) \qquad \text{in a coordinate basis.} \tag{7}$$

4. *Optional:* The covariant derivative of a contravariant vector U^μ is

$$\nabla_\nu U^\mu := \partial_\nu U^\mu + \Gamma^\mu_{\nu\lambda} U^\lambda\,. \tag{8}$$

Use this expression to obtain the covariant derivative of the covariant vector U_μ.

Lecture VIII

1. Consider the metric describing in polar coordinates (r, θ) an Euclidean space

$$ds^2 = dr^2 + r^2\, d\theta^2\,. \tag{1}$$

   - Compute the related Christoffel symbols and the geodesic curves associated to this space and given by the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0\,. \tag{2}$$

   - Combine the two second-order differential equations describing the geodesic curves into a single first-order differential equation for the polar curve $r = r(\theta)$. What does it describe?
   - What is the differential equation for a straight line in this space?

2. Consider the metric describing the two-dimensional spacetime covered by coordinates (t, x) and with metric

$$ds^2 = \frac{dx^2 - dt^2}{t^2}\,. \tag{3}$$

   - Compute the related Christoffel symbols.
   - Compute the geodesic curves of this space.

3. Given a scalar function $\phi = \phi(x^\mu)$, prove the following identity in a coordinate basis:

$$\Box\phi \coloneqq \nabla^\mu \nabla_\mu \phi = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} g^{\alpha\beta} \partial_\beta \phi \right)\,. \tag{4}$$

4. *Optional:* Derive the geodesic equation from the definition of a curve of extremal length.

Lecture IX

1. Consider a torus in a two-dimensional Euclidean space and in a spherical coordinate system (θ, ϕ). The line element is then

$$ds^2 = (b + a \sin \phi)^2 \, d\phi^2 + a^2 \, d\phi^2 \,, \tag{1}$$

   where b and a are the torus' radius and that of its section, respectively.

   Compute the Christoffel symbols and the nonvanishing components of the curvature tensor (Hint: remember there is only one linearly independent component of the Riemann tensor in a spacetime of dimensions 2).

2. Consider the two-dimensional spacetime with line element

$$ds^2 = dv^2 - v^2 du^2 \,. \tag{2}$$

   Compute the Christoffel symbols and the nonvanishing components of the curvature tensor.

3. *Optional:* Consider a geodesic curve $\mathscr{C}$ and its tangent vector $\boldsymbol{V}$. Compute the expression for the second convective derivative of a vector field $\boldsymbol{A}$ along $\mathscr{C}$, that is, the explicit expression in component form of

$$\nabla_{\boldsymbol{V}} \nabla_{\boldsymbol{V}} \boldsymbol{A} \,. \tag{3}$$

   Recast the resulting expression in terms of tensors that you have already encountered and interpret the results.

Lecture X

1. Show that the second covariant derivatives of a scalar field commute, *i.e.*, that

$$\nabla_\alpha \nabla_\beta \phi = \nabla_\beta \nabla_\alpha \phi \,. \tag{1}$$

   Obtain the expressions for the following third derivatives: $\nabla_\alpha \nabla_{(\beta} \nabla_{\gamma)} \phi$ and $\nabla_{[\alpha} \nabla_{\beta]} \nabla_\gamma \phi$ [Hint: remember that the covariant derivative of a scalar field is a vector].

2. Prove that for any second-rank tensor the covariant derivatives commute, *i.e.*, that

$$\nabla_\alpha \nabla_\beta V^{\alpha\beta} = \nabla_\beta \nabla_\alpha V^{\alpha\beta} \,. \tag{2}$$

3. *Optional:* Find the matrix of the Lorentz transformations corresponding to a boost v^x in the x direction followed by a boost v^y in the y direction. What happens if the order in the boost is inverted?

Lecture XI

All of the following exercises are to be considered in a special-relativistic context and assuming Cartesian coordinates when necessary.

1. Within Special Relativity, consider a four vector $\boldsymbol{V}$ with components:

$$\boldsymbol{V} = \sqrt{3}\,\boldsymbol{e}_t + \sqrt{2}\,\boldsymbol{e}_x\,. \tag{1}$$

   Determine if it is timelike, null or spacelike. Compute the angles between $\boldsymbol{V}$ and the unit vectors $\boldsymbol{e}_t$ and $\boldsymbol{e}_x$.

2. A particle with rest mass m and four-momentum $\boldsymbol{p} = m\boldsymbol{v}$ is analysed by an observer with four-velocity $\boldsymbol{u}$.

   - Compute the total energy E of the particle.
   - Compute the kinetic energy E_T of the particle.
   - Compute the magnitude of the spatial momentum $p := \sqrt{p^i p_i}$.
   - Compute the magnitude of the three velocity $v := \sqrt{v^i v_i}$.

3. Define the four-acceleration of a particle with four-velocity $\boldsymbol{u}$ as

$$a^\mu := \frac{du^\mu}{d\tau}\,, \tag{2}$$

   where τ is the proper time. Show that $\boldsymbol{a} \cdot \boldsymbol{u} = 0$, *i.e.*, the acceleration is orthogonal to the four-velocity. What does this mean in a frame comoving with the particle?

Lecture XII

1. Consider the energy-momentum tensor of a perfect fluid

$$T^{\mu\nu} = (e+p)u^\mu u^\nu + p g^{\mu\nu}\,, \tag{1}$$

   and its conservation equation

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{2}$$

   Show that the equations (2) lead to the Euler equations, *i.e.*, to the equations of conservation of momemntum

$$(e+p)\nabla_{\boldsymbol{u}}\boldsymbol{u} = -\left[\nabla p + (\nabla_{\boldsymbol{u}}\boldsymbol{u})\boldsymbol{u}\right]\,. \tag{3}$$

   [Hint: use the projector $\boldsymbol{h} = \boldsymbol{g} + \boldsymbol{u}\boldsymbol{u}$]. Do equations (3) bear resemblance with the Newtonian Eulerian equations?

2. The energy-mommentum tensor of a scalar field Φ is defined as

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\partial_\mu\Phi\,\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\,\partial_\alpha\Phi\partial^\alpha\,\Phi\right)\,. \tag{4}$$

   Derive the expression of the conservation of energy and momentum (2) in this case. Interpret the results.

3. Show that the Einstein equations in vacuum reduce to

$$R_{\mu\nu} = 0\,. \tag{5}$$

Lecture XIII

1. Consider the spherically symmetric static line element

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega^2\,, \tag{1}$$

and compute the expressions for the nonzero Christoffel symbols. Use the result to compute the 00 covariant component of the Einstein equations in vacuum $R_{\mu\nu} = 0$.

2. Using the Lagrangian

$$2\mathscr{L} = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\,, \tag{2}$$

where the dot stands for the derivative with respect to proper time, show that the geodesic equations

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0\,, \tag{3}$$

are equivalent to the Euler-Lagrange equations

$$\frac{\partial\mathscr{L}}{\partial x^\alpha} - \frac{d}{d\tau}\left(\frac{\partial\mathscr{L}}{\partial\dot{x}^\alpha}\right) = 0\,. \tag{4}$$

3. *Optional:* Using the Einstein-Hilbert action

$$\mathcal{S} = \int d^4\,\sqrt{-g}R\,, \tag{5}$$

show that the application of a variational principle $\delta\mathcal{S} = 0$ yields the Einstein field equations in vacuum, *i.e.*,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0\,. \tag{6}$$

Lecture XIV

1. Using the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right] , \tag{1}$$

   where $\kappa = -1, 0, 1$, compute

   - the nonzero Christffel symbols.
   - the nonzero components of the Ricci tensor.
   - the expression for the Ricci scalar.

2. Exploiting the results of the previous exercise, use the Einstein equations corresponding for the FRW metric, *i.e.*, the Friedmann equations. For simplicity set $\Lambda = 0$.

3. *Optional:* Consider the case of an equation of state in which $p = -e$ and $\Lambda > 0$. Derive the evolution equation for the scale factor. What type of Universe is this?

Lecture XV

1. The simplest solution to the linearized Einstein equations is a plane wave of the type
$$\bar{h}_{\mu\nu} = \Re\left\{A_{\mu\nu}\exp(i\kappa_\alpha x^\alpha)\right\}, \tag{1}$$
where $\Re$ selects the real part, $\boldsymbol{A}$ is the *"amplitude tensor"*, and $\boldsymbol{\kappa}$ is a null four-vector, *i.e.* $\kappa^\alpha \kappa_\alpha = 0$. In such a solution, the plane wave (1) travels in the spatial direction $\vec{k} = (\kappa_x, \kappa_y, \kappa_z)/\kappa^0$ with frequency $\omega \equiv \kappa^0 = (\kappa^j \kappa_j)^{1/2}$.

   Determine the conditions such that the amplitude tensor $A_{\mu\nu}$ has only 2 linearly independent components corresponding to the two states of polarization of the gravitational waves.

2. The gauge satisfying the requirement of first exercise is also referred to as the TT (or transvers-traceless) gauge. Compute the nonzero components of the Riemann tensor in this gauge.