Advanced General Relativity

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## Contents

1 Einstein field equations 5  
1.1 Equivalence principles ................................. 5  
1.2 Measure the curvature .................................. 6  
1.2.1 Covariant derivative ................................. 7  
1.2.2 Parallel transport and Riemann tensor ............... 8  
1.2.3 Geodesic deviation ................................... 11  
1.3 Einstein field equations ................................. 14  

2 Non-rotating black holes 16  
2.1 Schwarzschild coordinates ................................ 16  
2.1.1 Singularities .......................................... 17  
2.2 Eddington-Finkelstein coordinates ....................... 19  
2.3 Kruskal-Szekeres coordinates ............................ 21  
2.4 Carter-Penrose diagram .................................. 23  
2.5 Particle motion .......................................... 24  
2.6 Geodesic motion in effective potential .................. 26  
2.6.1 Circular motion ....................................... 28  
2.6.2 Impact parameter ..................................... 29  
2.6.3 Non-circular motion ................................... 30  
2.6.4 Local orthonormal tetrad .............................. 32  
2.6.5 Angular and radial velocity ............................ 33  
2.6.6 Massless particles ..................................... 34  

3 Rotating black holes 38  
3.1 Basic properties of the Kerr solution ..................... 38  
3.2 Singularities and horizons ................................ 40  
3.3 Geodesic motion in effective potential ................. 41  
3.3.1 Massive particles ..................................... 41  
3.3.2 Massless particles .................................... 42  
3.3.3 Penrose process ....................................... 43  

4 Relativistic stars 51  
4.1 Non-rotating stars ....................................... 51  
4.1.1 Conservation equations ................................ 51  
4.1.2 TOV equations ...................................... 52  
4.1.3 Gravitational mass and density profile .............. 53  
4.2 Rotating stars .......................................... 56  
4.2.1 Slow rotation approximation ......................... 56  
4.2.2 Mass-shedding limit .................................. 58  
4.3 Collapse of a dust sphere to a black hole .............. 60  
4.3.1 Trapped surface ...................................... 64  

5 Gravitational waves from perturbed black holes 68  
5.1 Linear perturbation of black holes ...................... 68  
5.2 Odd-parity perturbations: the Regge-Wheeler equation .... 70  
5.3 Even-parity perturbations: the Zerilli equation ........... 72  
5.4 QNMs of black holes ..................................... 74
6 3+1 splitting of spacetime

6.1 ADM formulation
6.1.1 3D formulation of basic elements
6.1.2 Extrinsic curvature
6.1.3 Decompose the Einstein equations
6.1.4 Constraint equations
6.2 ADM vs Maxwell
6.2.1 Hyperbolicity
6.3 BSSNOK formulation
6.4 CCZ4 formulation
6.5 Gauge conditions
6.5.1 Requirements of good gauge choices
6.6 Initial data
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- "Gravitation", Misner, Thorne, Wheeler (Freeman, 1973)
1 Einstein field equations

1.1 Equivalence principles

Special relativity relied on the existence of inertial frames, i.e. set of coordinates in which the separation among points does not change in space and the time intervals are the same everywhere.

\[ x^\mu = \Lambda^{\mu}_{\nu} x^\nu = \frac{\partial x^\mu}{\partial x'^\nu} x'^\nu \]

This construction is incompatible with the idea of gravity, which makes the existence of a global inertial frame impossible. Inertial frames are still allowed but only locally, i.e. at one point or its neighbourhood.

There are a number of ways to show that SR is incorrect in a gravitational field (e.g. redshift of photons, time dilatation, etc.), but let’s just take it as a fact: "there are no global inertial frame in a gravitational field".

Gravity makes global inertial frames impossible. To create such a frame, we need to remove gravity, i.e. we need to find a reference frame which is freely falling.

The two inertial frames are equivalent and both are inertial, at least locally.
They are inertial only in a position of spacetime. This is the **strong equivalence principle**: "the laws of physics in a free-falling frame are are the same as in an inertial frame" (i.e. as in SR). The strong equivalence principle is complementary to the **weak equivalence principle**, stating that it is not possible to distinguish between the action of gravity and that of an acceleration.

The "weight" in the two frames can be made to be exactly the same.

### 1.2 Measure the curvature

How do we reveal the existence of a gravitational field? A distinctive feature of **free** (not subject to a force) particles in SR is that they move on a straight line, so that two particles with parallel momenta will continue to move on parallel lines. Loss of inertial frame can be associated with a gravitational field, hence we can associate gravity with "**loss of parallelism**". On the other hand, loss of parallelism is also a natural property of a curved surface and so we can associate loss of parallelism with "**curvature**".

Stated differently, the answer to the question how do we reveal the existence of gravitational fields lies in the measurement of curvature. To this scope we need two different tools: the covariant derivative and parallel transport.
1.2.1 Covariant derivative

Take a four-vector \( V \) with \( V = V^\mu e^\mu \)

\[
\partial_\nu V = \partial_\nu (V^\mu e_\mu) = (\partial_\nu V^\mu) e_\mu + V^\mu \partial_\nu e_\mu
\]

we can express \( \partial_\nu e_\mu \) as another vector with components in the same basis

\[
\partial_\nu e_\mu = \Gamma^\lambda_{\mu\nu} e_\lambda
\]

which represent the change of the coordinates themselves between 2 points, so that

\[
\partial_\nu V = (\partial_\nu V^\mu) e_\mu + \Gamma^\lambda_{\mu\nu} e_\lambda
\]

\[
\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\lambda\nu} V^\lambda \quad (1)
\]

Note: this is not general relativity yet. All of this is just differential geometry. Also, covariant derivative is relevant even in a flat spacetime if the basis vectors are not constant.

Example 2D Cartesian coordinates

\[
ds^2 = g_{ij}dx^i dx^j = g_{xx} dx^2 + g_{yy} dy^2
\]

\[
g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g^{ij} = (g_{ij})^{-1}
\]

\[
V^\mu \partial_\mu e_\mu = V^j \partial_\mu e_\mu = 0
\]

Example 2D polar coordinates

\[
ds^2 = g_{ij} dx^i dx^j = g_{rr} dr^2 + g_{\theta\theta} d\theta^2
\]

\[
\begin{align*}
e_r \cdot e_r &= 1; & e_r \cdot e_\theta &= 0; & e_\theta \cdot e_\theta &= r^2 \\
\partial_r e_r &= 0; & \frac{\partial}{\partial \theta} e_r &= \frac{1}{r} e_\theta; & \frac{\partial}{\partial \theta} e_\theta &= -r e_r
\end{align*}
\]

In other words, the additional terms appearing in the covariant derivative (Christoffel symbols) may or may not reveal curvature.

How do we calculate the \( \Gamma \)'s? We need to use repeatedly the invariance of the metric tensor.

\[
\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} - \Gamma^\alpha_{\lambda\nu} g_{\alpha\mu} = 0
\]

\[
\nabla_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma^\alpha_{\nu\mu} g_{\alpha\lambda} - \Gamma^\alpha_{\nu\lambda} g_{\alpha\mu} = 0
\]

\[
\nabla_\nu g_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Gamma^\alpha_{\nu\lambda} g_{\alpha\mu} - \Gamma^\alpha_{\nu\mu} g_{\alpha\lambda} = 0
\]
\[ \nabla_\mu g_{\nu\lambda} + \nabla_\nu g_{\lambda\mu} - \nabla_\lambda g_{\mu\nu} = 0 = \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} - 2\Gamma^\alpha_{\nu\mu} g_{\lambda\alpha} \]

\[ \Rightarrow g_{\lambda\alpha} \Gamma^\alpha_{\nu\mu} = \frac{1}{2} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \]

Multiplying by \( g^{\beta\lambda} \)

\[ \Gamma^\beta_{\nu\mu} = \frac{1}{2} g^{\beta\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \quad (2) \]

### 1.2.2 Parallel transport and Riemann tensor

We need now to build a "curvature detector".

In a flat spacetime (e.g. blackboard) we can drag a vector along a closed curve (e.g. ABC) making sure that it always points in the same direction. This is called a "parallel transport" (note that the vector has to remain on the surface \( \Sigma \)). It’s not difficult to see that if we compare the vector before and after the closed curve they will be the same.

We can do the same on a curved surface, e.g. a 2-sphere \( \Sigma \)

The parallel transport has to keep the vector \( \mathbf{V} \) on \( \Sigma \) and always pointing in the same direction. Clearly, initial and parallel transported vector are different! Loss of parallel transport is due to (intrinsic) curvature in the surface \( \Sigma \). Note that there is the concept of **intrinsic** and **extrinsic** curvature but we will
discuss this later in the curse.

Let’s define parallel transport in a more rigorous way. Let $C(\tau)$ be a curve of parameter $\tau$ and $\mathbf{V}$ its tangent vector

$$\mathbf{V} = \frac{dx^\mu}{d\tau} e^\mu_i$$

Let $\mathbf{U}$ be another 4-vector; then the parallel transport of $\mathbf{U}$ along $C$ is

$$\nabla_\mathbf{V} \mathbf{U} = 0$$

or

$$V^\mu \nabla_\mu U^\nu = 0 \Leftrightarrow V^\mu \partial_\mu U^\nu + \Gamma^\nu_{\mu\alpha} U^\alpha V^\mu = 0$$

Note: a "straight" line is a curve that parallel transport its tangent vector.

$$\nabla_\mathbf{V} \mathbf{V} = 0 \Leftrightarrow V^\mu \nabla_\mu V^\nu = 0$$

This curve is also called geodesic and is strictly a straight line only in a flat spacetime.

We can now measure the curvature by parallel transporting a vector along two alternative paths and by comparing the difference, i.e.

$$R^\mu_{\nu\beta\alpha} V^\nu = 2 \nabla_\beta [\nabla_\alpha] V^\mu = -2 \nabla_\alpha [\nabla_\beta] V^\mu$$

where

$$R^\mu_{\nu\beta\alpha} = \partial_\beta \Gamma^\mu_{\alpha\nu} - \partial_\alpha \Gamma^\mu_{\beta\nu} + \Gamma^\mu_{\beta\delta} \Gamma^\delta_{\alpha\nu} - \Gamma^\mu_{\alpha\delta} \Gamma^\delta_{\beta\nu}$$

is the Riemann tensor.

Note:

- We know partial derivatives commute but this is not necessary the case for covariant derivatives, i.e. in general

$$\nabla_\alpha \nabla_\beta V^\mu \neq \nabla_\beta \nabla_\alpha V^\mu$$

- The Riemann tensor is a function of the second derivative of the metric or square of the first derivative of the metric, i.e.

$$R^\mu_{\nu\beta\alpha} = R^\mu_{\nu\beta\alpha}(\partial \Gamma, \Gamma \Gamma) = R^\mu_{\nu\beta\alpha}(\partial^2 g, (\partial g)^2)$$
Properties of the Riemann tensor

1. The Riemann tensor is antisymmetric on the last two indices

\[ R^{\mu}{}_{\nu\alpha\beta} = -R^{\mu}{}_{\nu\beta\alpha} \]

2. The tensor is also antisymmetric on the first two indices

\[ R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} \]

3. The tensor is symmetric on the exchange of the first and the second pair of indices

\[ R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \text{ or } R^{\mu}{}_{\alpha\beta\nu} = R^{\mu}{}_{\beta\nu\alpha} \]

4. The tensor is antisymmetric in the last three indices

\[ R^{\mu}{}_{[\alpha\beta\nu]} = R^{\mu}{}_{\alpha\beta\nu} + R^{\mu}{}_{\nu\alpha\beta} + R^{\mu}{}_{\beta\nu\alpha} = 0 \]

5. The tensor also satisfies the first Bianchi identity

\[ \nabla_{[\mu} R_{\nu\alpha\beta]\gamma} = \nabla_{\mu} R_{\nu\alpha\beta\gamma} + \nabla_{\alpha} R_{\mu\nu\beta\gamma} + \nabla_{\beta} R_{\alpha\mu\nu\gamma} = 0 \]  

(7)

6. Using the Riemann tensor and the Bianchi identity we can introduce two new tensors: the tensor obtained from the contraction of the first and the third index of the Riemann tensor, the Ricci tensor

\[ R^\alpha_{\beta\alpha\nu} = R_{\beta\nu} \]

and the Einstein tensor

\[ G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \]

which satisfies the Bianchi identity

\[ \nabla_{\beta} G^{\alpha\beta} = 0 \]

7. We can further contract the two indices of the Ricci tensor and obtain the Ricci scalar

\[ R := R^{\alpha}_{\alpha} = g^{\alpha\beta} R_{\beta\alpha} = R_{\alpha\beta} \]

(9)

and another scalar that can be built from the Riemann tensor, i.e. the Kretschmann scalar

\[ C := R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \]

(11)

What is the difference between the two scalars? Both of them concentrate on a single number all the information on the local curvature but have different dimensions.

\[ [g] = L^0 \text{ (dimension of the metric tensor)} \]
\[ [R^{\mu}{}_{\nu\alpha\beta}] = [\partial^2 g] = [(\partial g)^2] = L^{-2} \]
\[ \Rightarrow [R] = [R^{\alpha\beta}_{\alpha\beta}] = L^{-2} \]
\[ \Rightarrow [C] = [R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}] = L^{-2} L^{-2} = L^{-4} \]
1.2.3 Geodesic deviation

There is also another route to appreciate the relation between the curvature tensor and the presence of a gravitational field. We have discussed that a geodesic is a curve that parallel transport its tangent vector and that this is a straight line if the spacetime is flat. This is so because a geodesic is also the trajectory of a particle in free-fall. It is interesting therefore to consider the relative motion of two of such geodesics that are initially parallel, i.e. the geodesic deviation. We can consider this problem in Newtonian gravity first.

\[ r^a = r^a(t) \text{ along } C_1, \quad x^a = x^a(t) \text{ along } C_2 = r^a + \eta^a \]

where \( \eta \) is the vector measuring the separation between \( C_1 \) and \( C_2 \), with

\[ |\eta| \ll |r|, \quad |\eta| \ll |x| \]

(neighbouring particles).

Let \( \phi \) be the gravitational potential and the two particles freely falling, i.e.

\[ \ddot{r}^a = -\partial^a(\phi)_{P} \]
\[ \ddot{x}^a = -\partial^a(\phi)_{Q} = \ddot{r}^a + \ddot{\eta}^a = -\partial^a(\phi)_{P} + \ddot{\eta}^a \quad (12) \]

Expand the gravitational potential around \( P \), i.e.

\[ (\phi)_Q = (\phi)_P + \eta^b \partial_b(\phi)_P \]

so

\[ \ddot{x}^a = -\partial^a((\phi)_P + \eta^b \partial_b(\phi)_P) = -\partial^a(\phi)_P + \eta^b \partial^a \partial_b(\phi)_P \]
\[ = -\partial^a(\phi)_P - k^a_b \eta^b \quad (13) \]

where

\[ k^a_b = \partial^a \partial_b(\phi)_P \]

together with equation (12)

\[ \ddot{x}^a = -\partial^a(\phi)_P + \ddot{\eta}^a = -\partial^a(\phi)_P - k^a_b \eta^b \]

11
so that

\[ \ddot{\eta}^a = - k_a^b \dot{y}^b \]  

(14)

which is the equation of motion of the separation vector.

If the field is uniform \( \partial^a (\phi) P = 0 = \partial^a \partial_b (\phi) P = 0 \) and so

\[ \ddot{\eta}^a = 0 \Rightarrow \eta^a = \text{const} \]

Thus, also in Newtonian gravity two freely falling particles initially on parallel trajectories will intersect in a non uniform gravitational field (a gravitational field is necessarily non uniform!).

\[ \phi = \phi(r) = -\frac{M}{r}; \quad \partial^a \partial_b \phi = \partial^r \left( \frac{M}{r^2} \right) = -\frac{2M}{r^3} \]

and with that

\[ \ddot{\eta}^a \rightarrow \ddot{\eta}^r = -\frac{2M}{r^3} \]

shows that the tidal forces scale like \( r^{-3} \).

We can repeat the same analysis also in general relativity (GR)

where \( U_1 \) is the 4-vector tangent to the geodesic \( C_1(\tau, \nu_1) \), \( U_2 \) is the 4-vector tangent to the geodesic \( C_2(\tau, \nu_2) \) and \( \xi \) is the vector connecting the geodesic curves \( C_1 \) and \( C_2 \).

\[ \xi^\alpha = \frac{dx^\alpha}{d\nu} \]

where \( \nu \) is the parameter used to distinguish the geodesics. Each vector is parallel transported

\[ \nabla_{U_1} U_1 = 0 \]

Recalling the definition of Lie derivative \( \mathcal{L}_{U_1} \xi \)

\[ \mathcal{L}_{U_1} \xi = U^\beta \nabla_\beta \xi^\alpha - \xi^\beta \nabla_\beta U^\alpha \]

\[ = U^\beta \partial_\beta \xi^\alpha + \Gamma^\alpha_{\beta\mu} U^\beta \xi^\mu - \xi^\beta \partial_\beta U^\alpha - \Gamma^\alpha_{\beta\mu} \xi^\beta U^\mu \]

\[ = U^\beta \partial_\beta \xi^\alpha - \xi^\beta \partial_\beta U^\alpha + \Gamma^\beta_{\beta\mu} (U^\beta \xi^\mu - \xi^\beta U^\mu) \]

\[ = U^\beta \partial_\beta \xi^\alpha - \xi^\beta \partial_\beta U^\alpha \]
(which means that covariant derivatives reduce to partial derivatives) it’s not difficult to show that

$$\mathcal{L}_U \xi = 0 \Leftrightarrow \nabla_U \xi = \nabla_\xi U$$

(15)

Proof

$$\nabla_U \xi^\alpha = U^\beta \partial_\beta \xi^\alpha + \Gamma^\alpha_{\beta\mu} U^\beta \xi^\mu$$

$$\nabla_\xi U^\alpha = \xi^\beta \nabla_\beta U^\alpha = \xi^\beta \partial_\beta U^\alpha + \Gamma^\alpha_{\beta\mu} \xi^\mu$$

taking the difference

$$\nabla_U \xi^\alpha - \nabla_\xi U^\alpha = U^\beta \partial_\beta \xi^\alpha - \xi^\beta \nabla_\beta U^\alpha + \Gamma^\alpha_{\beta\mu} (U^\beta \xi^\mu - \xi^\beta U^\mu)$$

= $$\frac{dx^\beta}{dt} \frac{\partial \xi^\alpha}{d\nu} - \frac{dx^\beta}{d\nu} \frac{\partial \xi^\alpha}{dt} + \Gamma^\alpha_{\beta\mu} (U^\beta \xi^\mu - \xi^\beta U^\mu)$$

= $$\frac{d^2 x^\alpha}{d\tau d\nu} - \frac{d^2 x^\alpha}{d\nu d\tau} + \Gamma^\alpha_{\beta\mu} (U^\beta \xi^\mu - \xi^\beta U^\mu)$$

since \(\Gamma\) is symmetric on the lower indices and \(U_\xi\) are antisymmetric

$$\Gamma^\alpha_{\beta\mu} (U^\beta \xi^\mu - \xi^\beta U^\mu) = 0$$

and since

$$\frac{d^2 x^\alpha}{d\tau d\nu} - \frac{d^2 x^\alpha}{d\nu d\tau} = 0$$

it follows that the difference between \(\nabla_U \xi^\alpha\) and \(\nabla_\xi U^\alpha\) is zero. q.e.d.

Next we consider

$$2 \nabla_U \nabla_\xi U^\alpha = \nabla_U \nabla_\xi U^\alpha = 2 \nabla_U \nabla_\xi U^\alpha$$

(16)

Proof

$$2 \nabla_U \nabla_\xi U^\alpha = \nabla_U \nabla_\xi U^\alpha - \nabla_U \nabla_\xi U^\alpha = \nabla_U \nabla_\xi U^\alpha = \nabla_U \nabla_\xi U^\alpha$$

q.e.d.

The following identity can be proven for the Riemann tensor (exercise)

$$2 \nabla_{[X} \nabla_{Y]} Z^\alpha - \nabla_{[X} \nabla_{Y]} Z^\alpha = R^\alpha_{\beta\mu\nu} Z^\beta X^\mu Y^\nu$$

(17)

where

$$\nabla_{[X} \nabla_{Y]} Z^\alpha = \nabla_X (\nabla_Y Z^\alpha) - \nabla_Y (\nabla_X Z^\alpha)$$

and

$$\nabla_{[X} \nabla_{Y]} Z^\alpha = (\nabla_X Y)^\beta \partial_\beta Z^\alpha - (\nabla_Y X)^\beta \partial_\beta Z^\alpha$$

= $$X^\gamma \nabla_\gamma Y^\beta \partial_\beta Z^\alpha - Y^\gamma \nabla_\gamma X^\beta \partial_\beta Z^\alpha$$

If we now take \(X = Z = U\) and \(Y = \xi\) then the second term in equation (17) vanishes \((\nabla_U \xi) U^\alpha = 0\) since \(U\) and \(\xi\) commute \((\nabla_U \xi = \nabla_\xi U)\) and the general expression in (17) reduces to

$$2 \nabla_U \nabla_\xi U^\alpha = -R^\alpha_{\beta\mu\nu} U^\beta U^\mu \xi^\nu$$
which further reduces to
\[ \nabla_U (\nabla_U \xi^\alpha) = R^\alpha_{\beta\mu\nu} U^\beta U^\mu \xi^\nu \] (18)

with
\[ 2 \nabla_U \nabla_\xi U^\alpha = \nabla_U (\nabla_\xi U^\alpha) - \nabla_\xi (\nabla_U U^\alpha) = \nabla_U (\nabla_\xi U^\alpha) = \nabla_U (\nabla_U \xi^\alpha) = 0 \]

Equation (18) is called geodesic deviation equation and it can be further manipulated. The left-hand-side (LHS) can be written as
\[ \nabla_U (\nabla_U \xi^\alpha) = \frac{D^2}{D\tau^2} \xi^\alpha = \frac{d^2 \xi^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{d\xi^\mu}{d\tau} \frac{d\xi^\nu}{d\tau} \]

where
\[ \nabla_U = U^\alpha \frac{D}{D\tau} = \frac{dx^\alpha}{d\tau} \frac{D}{Dx^\alpha} = \frac{D}{D\tau} \]

The right-hand-side (RHS) can be also be rewritten as
\[ R^\alpha_{\beta\mu\nu} U^\beta U^\mu \xi^\nu = -R^\alpha_{\beta\mu\nu} U^\beta U^\mu \xi^\nu = -R^\alpha_{\beta\mu\nu} U^\beta \xi^\mu U^\nu = -k^\alpha_\mu \xi^\mu \]

where
\[ k^\alpha_\mu = R^\alpha_{\beta\mu\nu} U^\beta U^\nu \]

Putting this together
\[ \frac{D^2}{D\tau^2} \xi^\alpha = -k^\alpha_\mu \xi^\mu \]

(19)

We can now compare the Newtonian and the relativistic expressions
\[ \frac{d^2}{d\tau^2} \eta^\mu = -k^\alpha_\mu \eta^\mu = -\partial^a \partial_b (\phi) \eta^b \]
\[ \frac{D^2}{D\tau^2} \xi^\alpha = -k^\alpha_\mu \xi^\mu = -R^\alpha_{\mu\beta\nu} U^\mu U^\nu \xi^\beta \]

In this way we can appreciate that as in Newtonian gravity \( k^a_\beta = 0 \) when freely falling particles remain parallel, in GR \( R^\alpha_{\mu\beta\nu} = 0 \) when the geodesics remain parallel. By contrast, the convergence of the geodesics is a manifestation of non-zero curvature and hence of the presence of a gravitational field. This effect leads to “gravitational lensing” and is fully exploited in astronomy.

1.3 Einstein field equations

In Newtonian gravity we know that gravity is sourced by the presence of matter, e.g. as in the Poisson equation
\[ \nabla^2 \phi = 4\pi \rho \] (20)
Hence

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = k T_{\mu\nu} \]

where \( T_{\mu\nu} \) is the energy-momentum tensor (and the conservation of the energy and the momentum \( \nabla_\mu T_{\mu\nu} = 0 \) is a consequence of the Bianchi identity) and \( \Lambda \) is the cosmological constant. A bit of algebra (exercise) shows that

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}
\]

which are the Einstein field equations.

Taking the trace of equation (21), i.e. contracting with \( g^{\mu\nu} \)

\[
R_{\mu\mu} - \frac{1}{2} g^{\mu\mu} R + \Lambda g_{\mu\mu} G_{\mu\mu} = 8\pi g^{\mu\nu} T_{\mu\nu}
\]

\[
R - \frac{1}{2} 4R + 4\Lambda = 8\pi T
\]

\[ \Rightarrow R = -8\pi \left( T - \frac{\Lambda}{2\pi} \right) \]

so that

\[
R_{\mu\nu} = 8\pi \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T + \frac{\Lambda}{8\pi} g_{\mu\nu} \right)
\]

which is zero if \( T_{\mu\nu} = \Lambda = 0 \) (vacuum).

The Einstein equations are 10 non-linear 2nd-order partial differential equations in which the source produces geometry (\( T_{\mu\nu} \rightarrow G_{\mu\nu} \)) and, at the same time, the geometry prescribes the motion of the source (\( \nabla_\mu T_{\mu\nu} = 0 \)).
2 Non-rotating black holes

2.1 Schwarzschild coordinates

Let’s consider a static, spherically symmetric spacetime in vacuum. The most generic form of the line element for a spacetime of this type is

\[ ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2 \theta d\phi) \]

(23)

and under the assumption of vacuum, the Einstein equations

\[ R_{\mu\nu} = 8\pi \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \]

reduce to

\[ R_{\mu\nu} = 0 \]

(24)

Imposing asymptotic flatness it follows

\[ \lim_{r \to \infty} A = \lim_{r \to \infty} B = 1 \Leftrightarrow \text{const} = 1 \Leftrightarrow A = \frac{1}{B} \]

and the Einstein equations yield

\[ A = 1 + \frac{k}{r}, \quad B = \frac{1}{1 + \frac{k}{r}} \]

Matching with the weak-field limit yields

\[ g_{00} = -(1 + 2\Phi) = -\left(1 - \frac{2GM}{c^2 r}\right) = -\left(1 - \frac{2M}{r}\right) = -A = -\left(1 + \frac{k}{r}\right) \]

so that the Schwarzschild solution in Schwarzschild coordinates is

\[ ds^2 = -(1 - \frac{2M}{r})dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \]

(25)

Let’s consider some of the properties of this solution

• \( r \) is an area coordinate

\[ A = \int \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi = 4\pi r^2 \]

• \( r \) does not measure proper distances

\[ dl = \int_{r_A}^{r_B} \sqrt{g_{rr}} dr = \int_{r_A}^{r_B} \left(1 - \frac{2M}{r}\right)^{-1/2} dr \neq \int_{r_A}^{r_B} dr = r_{AB} \]
• $t$ is a time coordinate but does not measure the proper time. Consider a static observer ($dr = d\theta = d\phi = 0$)

$$ds^2 = -d\tau^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 \Rightarrow \frac{dt}{d\tau} = \left( 1 - \frac{2M}{r} \right)^{-1/2}$$

so that the ticking of clocks depends on position.

$$dt \to \infty \text{ for } r \to 2M$$
$$d\tau \to 0 \text{ for } r \to 2M$$

A direct consequence of this is the (gravitational) redshift of photons

$$\nu = \nu_0 \left( \frac{1 - \frac{2M}{r_0}}{1 - \frac{2M}{r}} \right)^{1/2} > 1$$

and

$$\nu \to 0 \text{ for } r_0 \to 2M$$

which is called infinity redshift.

• For $r \gg 1$, $r$ is spacelike ($ds^2 > 0$) and $t$ is timelike ($ds^2 < 0$); however for $r < 2M$ the opposite is true, radial coordinates become timelike and time coordinate becomes spacelike.

• In Newtonian gravity we are accustomed to the idea that gravity is sourced by mass, i.e. collection of baryons ($\nabla^2 \phi = 4\pi \rho$); this is a vacuum solution and yet generates a non-flat spacetime. The other vacuum spacetime solution is Minkowski and is flat.

### 2.1.1 Singularities

A useful way to measure properties of spacetime is to use scalars and the most revealing on is the Kretschmann scalar

$$C := R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{48M^2}{r^6} = \frac{3}{4M^2} \quad (26)$$
with $r = 2M$. The curvature is also singular only at $r = 0$; $r = 2M$ is not a physical singularity but only a coordinate singularity. Hence it can be removed via a suitable choice of coordinates. A good way of removing the coordinate singularity hinges on understanding what is the real pathology of the radial singularity $r = 2M$. We can do this by studying radial geodesics.

A simpler way of deriving the geodesic equations involves the Euler-Lagrange equations (exercise). By a variational principle we define the following Lagrangian

$$L = \int_{P_1}^{P_2} 2Ld\lambda = \int_{P_1}^{P_2} \sqrt{ds^2} = \int_{P_1}^{P_2} \sqrt{|g_{\alpha\beta}dx^\alpha dx^\beta|} = \int_{P_1}^{P_2} \sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}d\lambda$$

so that the condition (the extremal length curve)

$$\delta L = 0$$

leads to

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\lambda} \left( \frac{\partial L}{\partial \dot{x}^\alpha} \right) = 0 \quad (27)$$

where $\dot{x}^\alpha = dx^\alpha/d\tau$ ($\tau$ is the affine parameter) and $L$ is the Lagrangian given by

$$2L = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \begin{cases} -1 & \text{timelike tangent vector} \\ 0 & \text{null tangent vector} \\ +1 & \text{spacelike tangent vector} \end{cases}$$

In Schwarzschild

$$2L = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\left(1 - \frac{2M}{r}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \quad (28)$$

Let $\lambda = \tau$ be the proper time and consider a radial ($d\theta = d\phi = 0$) null geodesic ($2L = ds^2 = 0$)

$$2L = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\left(1 - \frac{2M}{r}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 = 0 \quad (29)$$

which gives in equation (27)

$$\frac{\partial L}{\partial x^0} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^0} \right) = 0 \Leftrightarrow \frac{d}{d\tau} \left[ \left(1 - \frac{2M}{r}\right)\dot{t} \right] = 0 \Leftrightarrow \left(1 - \frac{2M}{r}\right)\dot{t} = k = \text{const}$$

Using this condition, equation (29) becomes

$$\dot{r}^2 = k^2 \Leftrightarrow \dot{r} = \pm k$$

from which we deduce that also $r$ is an affine parameter. Equation (29) also implies that

$$\frac{\dot{t}}{\dot{r}} = \frac{dt/d\tau}{dr/d\tau} = \frac{dt}{dr} = \pm \frac{r}{r - 2M} \quad (30)$$
(plus for outgoing inputs and minus for ingoing inputs) which are worldlines of radial photons whose integration yields

\[ t = \pm (r + 2M \ln |r - 2M| + \text{const}) \]

Let’s study them in different regions of the spacetime

\[ \frac{dt}{dr} \to \pm 1 \quad \text{for} \quad r \to \infty \quad \text{(straight cones)} \]

\[ \frac{dt}{dr} \to \pm \infty \quad \text{for} \quad r \to 2M^\pm \quad \text{(cones have one side vertical)} \]

\[ \frac{dt}{dr} \to 0 \quad \text{for} \quad r \to 0 \quad \text{(cones are tilted)} \]

Note:

- Light cones tend to tilt as they are emitted near \( r = 2M \); exactly at \( r = 2M \) the cones are tangent at \( r = 2M \): photons can only go in and cannot go out.
- For \( r < 2M \) cones are completely tilted (t is spacelike and r is timelike)
- The singularity is in the future light cone of in and outgoing photons
- In this coordinates a photon takes an infinite coordinate time to reach \( r = 2M \)
- Spacetime is approximately flat at \( r \gg 1 \) and light cones are at 45°.

### 2.2 Eddington-Finkelstein coordinates

If the pathology at \( r = 2M \) is due to coordinates, we need to find better ones. In particular we need to find coordinates in which light cones are not distorted and are, for example, always at 45°. This can be achieved via a suitable coordinate transformation

\[ t \to t' = t + 2M \ln \left( \frac{r}{2M} - 1 \right) \]
For a radial photon we have seen
\[ t + r = -2M \ln \left( \frac{r}{2M} - 1 \right) + \text{const} \]
so that
\[ t' = -r + \text{const} \Leftrightarrow \frac{dt'}{dr} = -1 \]
meaning that the ingoing photons always have light cones at 45°. Because
\[ dt' = dt + \frac{2M}{r - 2M} dr \]
the line element is no longer diagonal and takes the form
\[
ds^2 = -\left(1 - \frac{2M}{r}\right)dt'^2 + \frac{4M}{r}dt'dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2d\Omega^2 \tag{32}\]
Not that there is no singularity at \( r = 2M \)! This is still the Schwarzschild solution but in the **ingoing Eddington-Finkelstein** coordinates.

Another coordinate transformation is possible in which one "straightens" the outgoing light cones, this leads to the **outgoing Eddington-Finkelstein** coordinates. Both descriptions are not singular at \( r = 2M \) but none of them is "maximal". We recall that "a manifold is said to be maximal if all geodesics emanating from any point in the manifold can be extended in all directions to infinite values of the affine parameter or terminate on a physical singularity". A maximal manifold without singularities is said to be "**geodesically complete**" (e.g. Minkowski).

There are several other sets of coordinates to express the Schwarzschild solution. What is relevant to remember is that coordinates are "just coordinates", i.e. only a choice out of the many possible ones. At the same time, a good choice makes the difference between being able to understand one solution or not (this has been the case of the Schwarzschild solution, that has been considered a mathematical solution without physical implication for decades).
2.3 Kruskal-Szekeres coordinates

A fully well-behaved and maximal coordinate system was derived in 1960 by Kruskal and Szekeres (KS), and provides the basis of our understanding of this solution (called black holes (BH) a few years later).

Consider the coordinate transformation

\[ \{t, r, \theta, \phi\} \rightarrow \{u, v, \theta, \phi\} \]

where there are two coordinate patches for \( r > 2M \) and \( r < 2M \). In particular, for \( r > 2M \)

\[
\begin{align*}
    u &= \left( \frac{r}{2M} - 1 \right)^{1/2} e^{r/4M} \cosh(t/4M) \\
    v &= \left( \frac{r}{2M} - 1 \right)^{1/2} e^{r/4M} \sinh(t/4M)
\end{align*}
\]

and for \( r < 2M \)

\[
\begin{align*}
    u &= \left( 1 - \frac{r}{2M} \right)^{1/2} e^{r/4M} \sinh(t/4M) \\
    v &= \left( 1 - \frac{r}{2M} \right)^{1/2} e^{r/4M} \cosh(t/4M)
\end{align*}
\]

with the line element

\[
ds^2 = -\frac{32M^3}{r} e^{-r/2M} (du^2 - dv^2) + r^2 d\Omega^2 \quad (33)
\]

(Note: \( u \) and \( v \) play now the role of the null coordinates in flat spacetime.)

Some properties of the KS coordinates are

- The metric is singular only at \( r = 0 \), while at \( r = 2M \) it is perfectly regular.
- Light cones are always opening at 45° everywhere

\[ ds^2 = 0 \Leftrightarrow du^2 = dv^2 \Leftrightarrow \frac{du}{dv} = \pm 1 \]

Each point of this diagram is a 2-sphere (\( \theta \) and \( \phi \) coordinates are suppressed).
• If \( t = \text{const} \), then we have straight lines through the origin

\[
\begin{aligned}
   t &= \begin{cases} 
4M \coth(u/v) & \text{in regions II and IV} \\
4M \coth(v/u) & \text{in regions I and III}
\end{cases}
\end{aligned}
\]

• If \( r = \text{const} \), then \( u^2 - v^2 = \text{const} \) which is the equation of an hyperbolas, which are timelike in region I and spacetime in region IV. \( r = \pm 2M \) represent the asymptotes of the hyperbolas and correspond to \( t = \infty \).

• The origin \((r = 0)\) is not a single point but a line (hyperbola). All \( t = \text{const} \) lines passes through such a line.

• \( r \) is an areal coordinate and the relation with \( u \) and \( v \) is given by

\[
\left( \frac{r}{2M} - 1 \right)e^{r/2M} = u^2 - v^2
\]

• In these coordinates a massive particle has timelike trajectory crossing curves of constant coordinate \( r \) and reaching \( r = 0 \) in finite time.

• The horizon \((r = 2M)\) is a null surface. Since photons move on \( 45^\circ \) lines, it is clear that a photon in region II cannot reach region I: they are causally disconnected.

• The four region of the Schwarzschild solution in KS coordinates have analogies with what already seen:
  
  - Region I and II: Ingoing Eddington-Finkelstein
  - Region II and III: Outgoing Eddington-Finkelstein
  - Region IV: Post singularity (white hole solution; photons emitted in region I cannot reach the post singularity)

Note: all those regions are causally disconnected, i.e. points in the two regions cannot be joined by photons (e.g. I and II, or I and III, or I and IV, ...).
2.4 Carter-Penrose diagram

The regularity of the KS coordinates can be combined with a conformal transformation bringing infinities at finite distance.

Consider $ds^2 = dt^2 + dx^2$ with

\[
\begin{align*}
  t &\to \tilde{t} = \frac{t}{1 + t} \\
  x &\to \tilde{x} = \frac{x}{1 + x}
\end{align*}
\]

where $J^+$ is the location of the endpoints of all outgoing photons and $J^-$ is the location of the origin of all ingoing photons. They are called null infinity. The same can be done also for the Schwarzschild solution and yields to the so-called Carter-Penrose diagram.
Essentially all of the properties seen for the KS coordinates continue to apply also to this diagram:

- $\mathcal{J}^+, \mathcal{J}^-$ are the $\pm$ null infinity
- $i^0$ is the spatial infinity
- $i^{\pm}$ is the future/past timelike infinity
- Photons still propagate along 45° cones
- There are 4 different regions (I, II, III, IV)
- Massive particles move as indicated

(We will get back to this diagram when discussing the 3 + 1 splitting and the different ways to "slice" spacetime)

2.5 Particle motion

Given a solution, it’s always useful to highlight its physical properties. In the case of the Schwarzschild solution consider the motion of a test particle (with no contribution to the RHS of the Einstein equations). From a physical point of view it is very similar to what you do in reality near a hole: throwing stones
is a way to sample properties of the hole (depth, width, etc.). Mathematically, we can have to solve the geodesic equations

\[ \nabla U = \frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0 \]

where \( U \) is the tangent vector of the geodesic curve.

Recalling the Lagrangian expressed in equation (28) we can show that

\[ 2L = g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = -\left(1 - \frac{2M}{r}\right) \dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = -m^2 \]

for a massive particle of mass \( m \).

**Proof**

With the definition of the conjugate variable to \( x^\alpha \)

\[ p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha} = g_{\alpha\beta} \dot{x}^\beta = g_{\alpha\beta} p^\beta \]

it follows

\[ p_\alpha p^\alpha = g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 2L = mU^\alpha U_\alpha = -m^2 \quad \text{q.e.d.} \]

A bit of algebra yields

\[ \frac{d}{d\lambda} (r^2 \dot{\theta}) = r^2 \sin \theta \cos \theta \dot{\phi}^2 \]  
\[ \frac{d}{d\lambda} (r^2 \sin^2 \theta \dot{\phi}) = 0 \]  
\[ \frac{d}{d\lambda} \left[ \left(1 - \frac{2M}{r}\right) \dot{t} \right] = 0 \]  
\[ \frac{d}{d\lambda} \left[ \left(1 - \frac{2M}{r}\right)^{-1} \dot{r} \right] = r (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \]

with \( \lambda = \tau/m \), where \( \tau \) is the proper time.

Equation (35) suggests us that the motion is planar.

**Proof**

Because of the imposed spherical symmetry we are free to choose \( \theta = \pi/2 \) so that

\[ \frac{d\dot{\theta}}{d\lambda} = 0 \quad \text{q.e.d.} \]

Equations (36) and (37) say us that \( t \) and \( \phi \) are ignorable coordinates and imply the existence of two constant of motion (exercise):

\[ p_\phi = \frac{\partial L}{\partial \dot{x}^\phi} = r^2 \sin^2 \theta \dot{\phi} = r^2 \dot{\phi} = l \]

which is the specific angular momentum, and

\[ p_t = \frac{\partial L}{\partial \dot{x}^t} = \left(1 - \frac{2M}{r}\right) \dot{t} = E \]

25
which is the energy at infinity.

Using equation (34) it is possible and easy to write the geodesic equations in a first-order form and obtain with \( \tilde{l} = l/m \) and \( \tilde{E} = E/m \):

\[
\begin{align*}
&\tilde{E} = E/m \left\{ \begin{array}{ll}
< 1 & : \text{rest at } R \\
1 & : \text{rest at } \infty \\
> 1 & : \text{positive velocity at } \infty
\end{array} \right.
\end{align*}
\]

\[
\frac{dr}{d\tau} = \pm \left[ \tilde{E}^2 - \left( 1 - \frac{2M}{r} \right) \left( 1 + \frac{\tilde{l}^2}{r^2} \right) \right]^{1/2} 
\]

\[
\frac{d\phi}{d\tau} = \frac{\tilde{l}}{r^2}
\]

\[
\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - 2M/r}
\]

Note: the geodesic equations do not depend on the mass of the particle.

Let’s first consider the simplest solution to equation (41), i.e. radial motion \((l = d\phi/d\tau = 0)\):

\[
\frac{dr}{d\tau} = - \left[ \tilde{E}^2 - \left( 1 - \frac{2M}{r} \right) \right]^{1/2}
\]

which yields for \( \tilde{E} < 1 \) to

\[
\tau = \left( \frac{R^3}{8M} \right)^{1/2} \left[ 2 \left( \frac{r - r_0^2}{R} \right)^{1/2} + \arccos \left( \frac{2r}{R} - 1 \right) \right]
\]

so that \( \tau = 0 \) for \( r = R \) and \( \tau = \pi \left( \frac{R^3}{8M} \right)^{1/2} \) for \( r = 0 \). In other words, the proper time needed to reach the singularity is always finite and proportional to \( R^{3/2} \); \( r = 2M \) is not a singular surface. This should be contrasted to the equation of motion in coordinate time

\[
\frac{dr}{dt} = \frac{dr/d\tau}{dt/d\tau} = - \frac{[\tilde{E} - (1 - 2M/r)]^{1/2}}{\tilde{E}/(1 - 2M/r)}
\]

for which

\[
t = - \frac{2[\tilde{l}^{3/2} - R^{3/2} + 6MR^{1/2}]}{3(2M)^{1/2}} + 2M \ln \left[ \frac{(r^{1/2} + (2M)^{1/2})(R^{1/2} - (2M)^{1/2})}{(r^{1/2} - (2M)^{1/2})(R^{1/2} + (2M)^{1/2})} \right]
\]

and so \( t \to \infty \) for \( r \to 2M \). In other words, the pathologies of the Schwarzschild coordinates become manifest also when computing geodesic dynamics; proper time does not diverge, although the coordinate time does.

### 2.6 Geodesic motion in effective potential

Let’s reconsider equation (41)

\[
\frac{dr}{d\tau} = \pm \left[ \tilde{E}^2 - \left( 1 - \frac{2M}{r} \right) \left( 1 + \frac{\tilde{l}^2}{r^2} \right) \right]^{1/2} = \pm [\tilde{E}^2 - V(r, \tilde{l})]^{1/2}
\]

26
where

$$V(r, \tilde{l}) = \left(1 - \frac{2M}{r}\right) \left(1 + \frac{\tilde{l}^2}{r^2}\right)$$

is the so-called "effective potential". Once $\tilde{l}$ is fixed, the effective potential leads to arises of well-defined orbits.

1. **Unbound hyperbolic orbit** $(\tilde{E}^2 > 1)$: the particle will move from spatial infinity, reach an inversion radius at $A$ such that $V = \tilde{E}^2$, $dr/d\tau = 0$ (inversion point) and return to spatial infinity.

2. **Captured orbit** $(\tilde{E}^2 > V \forall r)$: no turning point exists and the particle will fall onto the black hole ("captured").

3. **Bound elliptic orbit** $(\tilde{E}^2 < 1)$: two turning points exists and the particle will move between points $A_1$ and $A_2$.

4. **Bound circular orbit** $(\tilde{E}^2 < 1)$: at this radius the orbit is bound and stable, i.e. small perturbations will push the particle back at position B.

5. **Unstable circular orbit** $(\tilde{E}^2 > 1)$: this is a circular orbit but unstable, i.e. small perturbations will lead to capture or escape.

6. **Parabolic orbit** $(\tilde{E}^2 = 1)$: the particle has turning point at D and returns to spatial infinity with zero velocity.

Note: captured orbits are unique to GR. In Newtonian gravity the effective potential

$$V(r) = -\frac{GM}{r} + \frac{l^2}{2r^2}$$

so that
i.e. the potential is always able to repel you.

2.6.1 Circular motion

Circular orbits exist for $\partial_r V|_{\tilde{l} = \text{const}} = 0$, which means

$$Mr^2 - \tilde{l}^2 r + 3M\tilde{l}^2 = 0$$

Distinct minima and maxima exist for $\tilde{l} \geq 2\sqrt{3}M$ (exercise). At circular orbits then

$$\tilde{l}^2 = \frac{Mr^2}{r - 3M}$$

and

$$\tilde{E}^2 = \frac{(r - 2M)^2}{r(r - 3M)}$$

Stability is obtained also with the additional condition $\partial_r^2 V|_{\tilde{l} = \text{const}} \geq 0$, which implies

$$2Mr - \tilde{l}^2 \geq 0$$

so that $r \geq 6M$. $r = 6M$ is then the innermost stable circular orbit (ISCO) and is obtained for $\tilde{l} = 2\sqrt{3}M$. Stable and unstable circular orbits merge (as clear in the picture below).

Note:

- $V_{\text{max}}$, i.e. the value of $V$ for the unstable circular orbit, is equal to 1 for $r_{mb} = 4M$ so that also $\tilde{E} = 1$. $r_{mb}$ defines the so-called marginally bound orbit. This is the parabolic orbit that penetrates further in the effective potential and is unbound.
• Bound stable circular orbits exists from

\[ 6M \leq r < \infty \Leftrightarrow 2\sqrt{3}M \leq \tilde{l} < \infty \]

A large angular momentum allows stable circular orbits at large distances. The energy drop between spatial infinity and the ISCO is

\[
-E_b = \frac{\Delta E}{E_\infty} = \frac{E_{ISCO}}{E_\infty} - 1 = E_{ISCO} - 1 = \left[ \frac{(r - 2M)^2}{r(r - 3M)} \right]_{r=6M}^{r=\infty} - 1
\]

\[ = \left( \frac{8}{5} \right)^{1/2} - 1 \approx 0.057 \]

This 6% has to be compared with the 0.9% of nuclear fusion. It means that accretion into a black hole is a very efficient way of extracting energy.

In the case of a circular orbit it is also possible to calculate the angular velocity as measured from an observer at infinity

\[
\Omega = \frac{\dot{\phi}}{\dot{t}} = \frac{\tilde{l}}{r^2} \frac{1 - 2M/r}{E} = \left( \frac{r - 2M}{r^3} \right) \frac{\tilde{l}}{E} = \left( \frac{r - 2M}{r^3} \right) \frac{\sqrt{Mr}}{1 - 2M/r} = \left( \frac{M}{r^3} \right)^{1/2}
\]

which is the Keplerian angular velocity.

2.6.2 Impact parameter

Let’s now consider the impact parameter for a massive particle flying by the BH; \( b_{\text{max}} \) is the largest impact parameter

\[
b_{\text{max}} = \lim_{r \to \infty} r \sin \phi
\]

\( r \) diverges, \( \sin \phi \) goes to zero, but their product must be non-zero for a particle non moving on radial motion).

Using equation (41) and (42) we obtain

\[
\left( \frac{\dot{r}}{\dot{\phi}} \right)^2 = \left( \frac{dr}{d\phi} \right)^2 = r^4 \frac{[\tilde{E}^2 - (1 - 2M/r)(1 + \tilde{l}^2/r^2)]}{\tilde{l}^2} \quad (44)
\]

29
so that for $r \to \infty$
\[
\frac{1}{r^4} \left( \frac{dr}{d\phi} \right)^2 = \frac{\tilde{E}^2 - 1}{l^2}
\]  
(45)

For a particle an near radial geodesics, i.e. $\phi \ll 1$, $b \approx r\phi$ or $dr/d\phi = -b/\phi^2$ so that from equation (45) we obtain
\[
\frac{1}{r^4} \frac{b^2}{\phi^4} = \frac{1}{b^2} = \frac{\tilde{E}^2 - 1}{l^2} = \frac{v_{\infty}^2}{l^2(1 - v_{\infty}^2)}
\]

with
\[
\tilde{E}^2 = \frac{1}{1 - v_{\infty}^2}
\]

so that in the end
\[
\tilde{l} = \frac{b v_{\infty}}{\sqrt{1 - v_{\infty}^2}} \approx b v_{\infty}
\]  
(46)

In other words, for a not relativistic particle at $\infty$ (i.e. $v_{\infty} \ll 1$ and $\tilde{E} \approx 1$) the capture occurs for $\tilde{l} \geq 4M$ ($v_{\text{max}} = 1$ for $\tilde{l} = 4M$ so that larger values lead to capture) or $4M = bv_{\infty}$

\[
b_{\text{max}} = \frac{4M}{v_{\infty}}
\]

The corresponding capture cross section is
\[
\sigma_{\text{max}} = \pi b_{\text{max}}^2 = \frac{16\pi M}{v_{\infty}^2}
\]  
(47)

Let’s compare equation (47) with the corresponding Newtonian expression for a gravitating mass $M$ of radius $R$
\[
\sigma_{\text{Newton}} = \pi R^2 \left(1 + \frac{2M}{v_{\infty}^2 R} \right) = \pi R^2 \left( v_{\infty}^2 + \frac{2M}{R} \right) \approx \frac{2\pi MR}{v_{\infty}^2} = \frac{4\pi M^2}{v_{\infty}^2}
\]

with $R = 2M$. It follows that $\sigma_{\text{max}} \approx 4\sigma_{\text{Newton}}$: a BH has a cross section which is four times larger, or a radius twice as big. This a typical characteristic of GR effects: gravity is "stronger".

### 2.6.3 Non-circular motion

Let’s consider a bound non-circular orbit, i.e. $\tilde{E} < 1$ and $\partial_r V \neq 0$. Using equation (44) and defining $U = 1/r$ so that
\[
\left( \frac{dr}{d\phi} \right)^2 = \frac{1}{U^4} \left( \frac{dU}{d\phi} \right)^2
\]
we obtain
\[
\left( \frac{dU}{d\phi} \right)^2 = \tilde{E}^2 - \frac{1}{l^2} - \left( 1 - 2MU \right) \left( \frac{1}{l^2} + U^2 \right)
\]  
(48)
Let’s consider orbits that are \textbf{quasi-circular} and we can do this by working perturbatively around circular orbits, i.e. orbits for which

\[
\partial_r V = 0 \Leftrightarrow r = \frac{l^2}{2M}\left(1 \pm \sqrt{1 - \frac{12M^2}{l^2}}\right) \approx \frac{l^2}{2M}(1 + \epsilon) \approx \frac{2l^2}{2M}
\]

We introduce a parameter \( y = \frac{1}{r} - \frac{M}{l^2} = U - \frac{M}{l^2} \ll 1 \) for quasi-circular orbits and rewrite (48) as

\[
\left(\frac{dy}{d\phi}\right)^2 \approx \frac{\tilde{E}^2 + M^2/l^2 - 1}{l^2} + \frac{2M^4}{l^6} + \frac{6M^3}{l^2}y + \left(\frac{6M^2}{l^2}\right)y^2 + O(y^3)
\]

whose solution is

\[
y = y_0 + A \cos(k\phi + \phi_0)
\]

where

\[
y_0 = \frac{3M^2}{k^{1/2}}; \quad k = \left(1 - \frac{6M^2}{l^2}\right)^{1/2}; \quad A = \frac{1}{k}\left(\tilde{E}^2 + \frac{M^2}{l^2} - 1 + \frac{2M^4}{l^6} - y_0^2\right)^{1/2}
\]

This solution is periodic around \( y_0 \), which however is not a constant as in Newtonian gravity, rather \( y_0 = y_0(\tilde{l}) \).

We recall that the Newtonian solution is

\[
U = \tilde{k}(1 + \epsilon \cos(\phi - \phi_0)) \Leftrightarrow r = \frac{\tilde{k}^{-1}}{1 + \epsilon \cos(\phi - \phi_0)}
\]

(\( \epsilon \) is the eccentricity of the system).

Most importantly, \( k \neq 1 \), i.e. after a revolution (\( \phi \to 0 \) or \( \phi \)) the orbit does not close, but the periastron precesses of an angle

\[
\Delta \phi = \frac{2\pi}{k} = 2\pi \left(1 - \frac{6M^2}{l^2}\right)^{-1/2} \approx 2\pi \left(1 + \frac{3M^2}{l^2}\right)
\]

The periastron shift per orbit is then

\[
\Delta \phi = \frac{6\pi M^2}{l^2}
\]

(49)

The measurement of this shift for Mercury solved a fundamental problem in celestial mechanics and is a standard measure in binary pulsars.
2.6.4 Local orthonormal tetrad

Before moving to massless particles (i.e. photons) it is useful to consider the properties of the motion for observers that are static, e.g. near the black hole horizon. To this scope we need to introduce a local orthonormal tetrad, i.e.

\[ \{e_\alpha^\mu\} = \{(e_1^t, e_2^r, e_3^\theta, e_4^\phi); (e_1^r, e_2^\theta, e_3^\phi, e_4^t); (e_1^\theta, e_2^t, e_3^r, e_4^\phi); (e_1^\phi, e_2^t, e_3^r, e_4^\theta)\} \]

such that

\[ \xi_\alpha \xi_\beta = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Note that the standard tetrad in Schwarzschild coordinates is orthogonal but not normal. Because the metric in this coordinates behaves as if in flat space-time, the measurements made in this frame are "proper" measurements, i.e. measurements as made in the absence of gravity (gravity is cancelled via some artificial acceleration). In this respect they have a simple Newtonian interpretation.

In Schwarzschild coordinates, the coordinate basis \( \xi_\alpha \) is composed of 4 four-vectors \( \xi_t, \xi_r, \xi_\theta \) and \( \xi_\phi \), where \( \xi_t = (1, 0, 0, 0) \), \( \xi_r = (0, 1, 0, 0) \), \( \xi_\theta = (0, 0, 1, 0) \) and \( \xi_\phi = (0, 0, 0, 1) \), i.e. \( \xi_\mu^\mu = \delta_\alpha^\alpha \). This basis is orthogonal but not normal.

Example

\[ \xi_t \cdot \xi_t = g_{t\alpha} \xi_\alpha^\alpha \xi_\alpha^\beta = g_{tt} = \left( 1 - \frac{2M}{r} \right)^{-1} \neq 1 \]

The corresponding orthonormal basis and 1-form are (exercise)

\[
\begin{align*}
\xi_t & = \left( 1 - \frac{2M}{r} \right)^{-1/2} \xi_t \\
\xi_r & = \left( 1 - \frac{2M}{r} \right)^{1/2} \xi_r \\
\xi_\theta & = \frac{1}{r} \xi_\theta \\
\xi_\phi & = \frac{1}{r \sin \theta} \xi_\phi
\end{align*}
\]

(50)

\[
\begin{align*}
\tilde{\omega}^t & = \left( 1 - \frac{2M}{r} \right)^{1/2} \tilde{\omega}^t \\
\tilde{\omega}^r & = \left( 1 - \frac{2M}{r} \right)^{-1/2} \tilde{\omega}^r \\
\tilde{\omega}^\theta & = r \tilde{\omega}^\theta \\
\tilde{\omega}^\phi & = r \sin \theta \tilde{\omega}^\phi
\end{align*}
\]

(51)

It’s then straightforward to verify that, for instance,

\[ \xi_t \cdot \xi_t = g_{t\alpha} \xi_\alpha^\alpha \xi_\alpha^\beta = g_{tt} \xi_t^\mu \xi_t^\mu = \left( 1 - \frac{2M}{r} \right) \left( 1 - \frac{2M}{r} \right)^{-1} = -1 = \eta_{tt} \]
or
\[ \mathbb{L}_\rho \cdot \mathbb{L}_\tau = g_{\alpha \beta} e^\alpha_\rho e^\beta_\tau = g_{rr} \left( 1 - \frac{2M}{r} \right) = 1 = \eta_{\rho \tau} \]

2.6.5 Angular and radial velocity

We can now ask questions like: what is the energy measured by such a static observer? What’s the velocity of a geodesic particle? Recall that in SR the locally measured energy (the energy measured in a frame in which the particle is at rest) is
\[ E = -pu = -mU^\alpha U_\alpha = m \]

where \( m \) is the rest mass. In other words \( -p_0 = m \) is the energy of a particle in a frame in which it is at rest. The energy measured by a local static observer will then be
\[ E_{\text{local}} = -pu = -p_\alpha U^\alpha = -p_\alpha e^\alpha_t = -p_t \left( 1 - \frac{2M}{r} \right)^{-1/2} e_t \]

\[ = p_t \left( 1 - \frac{2M}{r} \right)^{-1/2} = E \left( 1 - \frac{2M}{r} \right)^{-1/2} > E \]

where \( e^\alpha_t = \delta^\alpha_t \) and we defined \( E = -p_0 \), so that

for \( r \to \infty \), \( E_{\text{local}} = E \)

for \( r \to 2M \), \( E_{\text{local}} \to \infty \)

It than useful to notice that
\[ \frac{E_\infty}{E_{\text{local}}} = \frac{E}{E_{\text{local}}} = \frac{\nu_\infty}{\nu} = \left( 1 - \frac{2M}{r} \right)^{1/2} \]

so that

for \( r \to \infty \), \( \frac{\nu_\infty}{\nu} \to 1 \) : no redshift

for \( r \to 2M \), \( \frac{\nu_\infty}{\nu} \to \infty \) : infinite redshift

In this frame it is simple to compute, for instance, the angular and radial velocity of a particle on a geodesic orbit. For example
\[ v^\phi = \frac{p^\phi}{p^t} = \frac{p_\alpha e^\alpha_\phi}{p_\alpha e^\alpha_t} = \frac{p_\phi e^\phi_\phi}{p_\phi e^\phi_t} = \frac{p_\phi/r \sin \theta}{E(1 - 2M/r)^{-1/2}} = \frac{l}{r \sin \theta E_{\text{local}}} \]

so that we come to definition
\[ l = v^\phi r \sin \theta E_{\text{local}} \]

which compared to the Newtonian definition
\[ l = mv^\phi r \sin \theta \]
confirms the interpretation of $l$ as a specific angular momentum. Note also that

$$v^\phi = \frac{l(1 - 2M/r)^{1/2}}{r \sin \theta E} = 0$$

(56)

for $r \to 2M$. It means that all particles enter the horizon with zero angular velocity.

Similarly

$$v^r = \frac{p^r}{p^t} = \frac{p_r}{E_{\text{local}}} = \frac{p_r}{E} \left(1 - \frac{2M}{r}\right) = \frac{\dot{r}}{E}$$

We have seen that $p_r = m d\tau/d\tau$, and using the radial geodesic equation we obtain

$$v^r = \frac{1}{E} \left[1 - \frac{1}{E^2} \left(1 - \frac{2M}{r}\right) \left(1 + \frac{\dot{l}^2}{r^2}\right)\right] = 1$$

(57)

for $r \to 2M$ (this is true for any finite $\tilde{E}$ and $\tilde{l}$). The result means that all particles enter the horizon at the speed of light, independently of the energy and angular momentum.

### 2.6.6 Massless particles

Let’s consider now massless particles: photons. In this case $m = 0$ and $\tilde{E}$ and $\tilde{l}$ are not defined. We can however use $E = -p_0$ and $l = p_\phi$ as constant of motion. In this case, the geodesic equations are again obtained from the Euler-Lagrangian equations for $2L = -m^2 = 0$. The resulting equations are

$$\dot{t} = \frac{dt}{d\lambda} = \frac{E}{1 - 2M/r}$$

$$\dot{\phi} = \frac{l}{r^2}$$

$$(\dot{r})^2 = E^2 - \left(1 - \frac{2M}{r}\right) \frac{l^2}{r^2}$$

The problem with these geodesic equations is that they are dependent on the energy of the photon and this is in contrast with the equivalence principle (all photons should be subject to the same acceleration). Started differently, as long as photons are test particles, they will have to have the same geodesic at all energies. To solve this problem, we can use another affine parameter

$$\lambda \to \tilde{\lambda} = l \lambda$$

so that

$$\frac{d}{d\tilde{\lambda}} = \frac{1}{l} \frac{d}{d\lambda}$$
and geodesic equations become
\[ \dot{t} = \frac{1}{b(1 - 2M/r)} \] (58)
\[ \dot{\phi} = \frac{1}{r} \] (59)
\[ (\dot{r})^2 = \frac{1}{b^2} - \frac{1}{r^2} \left(1 - \frac{2M}{r}\right) = \frac{1}{b^2} - V_{ph} \] (60)
where \( b := l/E \) is the impact parameter and
\[ V_{ph} = \frac{1}{r^2} \left(1 - \frac{2M}{r}\right) \] (61)
is the effective potential. Circular orbits at the maximum of the potential are given by
\[ \partial_r V_{ph} = 0 \iff \partial_r \left[ \frac{1}{r^2} \left(1 - \frac{2M}{r}\right) \right] = 0 \iff r = 3M \] (62)
so that we can define the critical value of the impact parameter \( b_c \) as
\[ V_{ph}|_{3M} = \frac{1}{27M^2} \Rightarrow b_c^2 = \frac{1}{V_{ph}} = 27M^2 \iff b_c = 3\sqrt{3}M \] (63)
and the photon cross section will be
\[ \Rightarrow \sigma_{ph} = \pi b_c^2 = 27\pi M^2 \] (64)

Only two orbits are possible: captured orbit \((b < b_c)\) and escape orbit \((b > b_c)\). A third orbit, by \( r = 3M \), leads to an unstable circular orbit, a "light ring". Computing the motion of photons in a spacetime is usually referred to as "ray tracing" and requires the solution of the geodesic equations. However, it’s easy to estimate some simple expressions to understand whether a photon will be captured if we use again the orthonormal basis. The static observer will see the photon reach infinity if \( v^r > 0 \) (outgoing photon) or \( v^r < 0 \) but \( b > 3\sqrt{3}M \) (ingoing photon).
Let $\Psi$ be the angle between the direction of propagation and the radial direction, then

$$v^\alpha v_\alpha = 1 = (v^r)^2 + (v^\phi)^2 = \sin^2 \Psi + \cos^2 \Psi$$

with $v^r = \sin \Psi$ and $v^\phi = \cos \Psi$.

We have seen that

$$v^\phi = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{l}{rE} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{1}{b}$$

Hence, an ingoing photon ($v^r < 0$) will escape to infinity if emitted with an angle

$$\sin \Psi > \left(1 - \frac{2M}{r}\right)^{1/2} \frac{1}{b} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{3\sqrt{3}M}{r}$$

and conversely, an outgoing photon emitted at $r/M \in [2, 3]$ will escape to infinity if

$$\sin \Psi < \left(1 - \frac{2M}{r}\right)^{1/2} \frac{1}{b} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{3\sqrt{3}M}{r}$$

**Example**  Ingoing

$$r = 6M \Leftrightarrow \sin \Psi > \frac{\sqrt{2}}{2} \Leftrightarrow \frac{3}{4} \pi < \Psi < \frac{\pi}{4}$$

$$r = 3M \Leftrightarrow \sin \Psi = 1 \Leftrightarrow \Psi = \frac{\pi}{2}$$

$$r = 0 \Leftrightarrow \sin \Psi = 0 \Leftrightarrow \text{no escape}$$

**Example**  Outgoing

$$r = 6M \Leftrightarrow \sin \Psi < \frac{\sqrt{2}}{2}$$

36
3 Rotating black holes

Schwarzschild solution was found in 1916 and a charged black hole solution (Reissner-Nordström) was found soon afterwards (1916-1918). It is necessary to wait until 1963 for Kerr to derive the rotating solution. In Boyer-Lindquist coordinates it reads

\[
\begin{aligned}
    ds^2 &= -\left(1 - \frac{2Mr}{\Sigma^2}\right)dt^2 - \frac{4\alpha Mr \sin^2 \theta}{\Sigma^2} dtd\phi + \frac{\Sigma^2}{\Delta} dr^2 + \Sigma^2 d\theta^2 + \frac{A}{\Sigma^2 \sin^2 \theta} d\phi^2
    \\
\end{aligned}
\]

(65)

where

\[
\begin{align*}
    \Delta &:= r^2 - 2Mr + a^2; \quad A := (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta; \\
    \Sigma^2 &:= r^2 + a^2 \cos^2 \theta; \quad a := \frac{J}{M}
\end{align*}
\]

This is an incredibly rich solution and a whole course could be dedicated to its discussion. We will here concentrate only on the most important aspects.

3.1 Basic properties of the Kerr solution

The first of these aspects is the existence of an axial Killing vector. We recall that a vector field \( \xi^\mu \) is said to be a Killing field if the metric is Lie translated along the congruence of \( \xi \), i.e. if and only if

\[
    \mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu \xi^\alpha + g_{\nu\alpha} \partial_\mu \xi^\alpha = 0
\]

(66)

A direct consequence of this equation is that \( \xi \) satisfies to Killing equation

\[
    \nabla_\mu \xi_\nu = 0
\]

(67)

Equations (67) determine the isometries of the spacetime. When the Killing vector coincides with a basis vector (and it is always possible to find the coordinate system where this happens) then (66) shows that the corresponding coordinate is cyclic, i.e. the metric doesn’t depend on it.

Example In the case of the Kerr solution there are two Killing vectors \( \eta^\mu = \delta^\mu_t \) and \( \xi^\mu = \delta^\mu_\phi \) such that

\[
    \eta^\mu \eta_\mu = g_{tt}; \quad \eta^\mu \xi_\mu = g_{t\phi}; \quad \xi^\mu \xi_\mu = g_{\phi\phi}
\]

So, an important consequence of the existence of the \( \xi^\mu \) Killing vector is the presence of a non diagonal \( t - \phi \) metric component. In Boyer-Lindquist coordinates this is given by

\[
    g_{t\phi} = -\frac{2aMr \sin \theta}{\Sigma} = -\frac{2aMr \sin \theta}{r^2 + a^2 \cos^2 \theta}
\]
and it is responsible for an interesting phenomenon. Consider a particle with momentum

\[ p^\phi = g^{\phi\alpha} p_\alpha = g^{\phi t} p_t + g^{\phi\phi} p_\phi = m \frac{d\phi}{d\tau} \]

\[ p^t = g^{t\alpha} p_\alpha = g^{tt} p_t + g^{t\phi} p_\phi = m \frac{dt}{d\tau} \]

The angular velocity will be

\[ \Omega := \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = p^\phi = \frac{g^{\phi t} p_t + g^{\phi\phi} p_\phi}{g^{tt} p_t + g^{t\phi} p_\phi} = \frac{g^{\phi t}}{g^{tt}} \] (68)

with the assumption of a particle with zero angular momentum, i.e. \[ p_\phi = 0 \].

Summarizing we have

\[ \Omega = \frac{g^{\phi t}}{g^{tt}} = \omega \] (69)

which says us that the particle has an angular velocity \( \omega(r, \theta) \) even if moving radially initially.

We define also the Lense-Thirring angular velocity

\[ \omega(r, \theta) = \frac{2Ma}{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta} \] (70)

Note: \( \omega(r, \theta) \propto \frac{a}{r} \) is the manifestation of the "dragging" of inertial frames. To appreciate this dragging, let’s consider a generic particle four-velocity \( u \), such that

\[ u \cdot u = -1 = (ut)^2 (g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}) \]

Imposing that the content of the round brackets is negative, we define

\[ \Omega_{\min} < \Omega < \Omega_{\max} \]

where

\[ \Omega_{\min,\max} = -\frac{g_{t\phi} \pm (g_{t\phi} - g_{tt}g_{\phi\phi})^{1/2}}{g_{\phi\phi}} \]

\[ = \omega_\phi (\omega^2 - g_{tt}/g_{\phi\phi})^{1/2} = \pm \frac{1}{r} \left( 1 - \frac{2M}{r} \right)^{1/2} \] (71)
for $a = 0$, where $+$ leads to $\Omega_{\text{min}} \neq 0$ and $-$ leads to $\Omega_{\text{max}}$.

Started differently, for any massive particle moving on a timelike geodesic ($u \cdot u = -1$) the angular velocity is not arbitrary; because $\Omega_{\text{min}} \neq 0$ there are regions of spacetime where no static particle can exist.

3.2 Singularities and horizons

In the Schwarzschild solution the event horizon is defined as the surface at which

$$g_{tt} = 0 \iff g_{rr} = \infty$$

In Kerr, the condition $g_{rr} = \infty$ occurs when $\Delta = 0$ or equivalently when

$$r = r_\pm = M \pm (M^2 - a^2)^{1/2}$$

(72)

where $r_- < r_+$, while the condition $g_{tt} = 0$ occurs when

$$r = r_0 = M + (M^2 - a^2 \cos^2 \theta)^{1/2} > r_+$$

(73)

**Note:** $r_0 = 2M$ for any $a$ on equatorial plane, i.e. $\theta = \pi/2$. Since $g_{tt} = 0$ at $r_0$, $\Omega_{\text{min}} = 0$ there; $r_0(r, \theta)$ marks the "static" limit. All particles must co-rotate for $r \leq r_0$.

As in the Schwarzschild solution in Schwarzschild coordinates, the Kerr solution in Boyer-Lindquist coordinates is singular at the event horizon. However, also in this case it is possible to find coordinate systems, e.g. the Kerr-Schild coordinates, that are not singular at the horizon. The only physical singularity of the Kerr solution in Kerr-Schild coordinates takes place at $\Sigma^2 = 0 = r^2 + a^2 \cos^2 \theta$, that is for $r = 0$ and $\theta = \pi/2$. Note that this seems to imply that $r = 0$ and $\theta \neq \pi/2$ is not a singularity. This is just a poor behaviour of the spherical coordinates near $r = 0$. In Kerr-Schild Cartesian coordinates for $\theta = \pi/2$

$$\Sigma^2 = 0 \iff \frac{x^2 + y^2}{r^2 + a^2} = 1 \iff x^2 + y^2 = r^2 + a^2 = a^2$$

for $r = 0$, which is the so-called "ring singularity".
3.3 Geodesic motion in effective potential

3.3.1 Massive particles

Let’s first consider massive particles in the equatorial plane, i.e $\theta = \pi/2$ (note that particle motion is no longer planar in a Kerr spacetime)

$$2L = -\left(1 - \frac{2M}{r}\right)i^2 - 4aM \frac{i}{r} + \frac{r^2}{\Delta} \dot{r}^2 + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi}^2 = -1 \quad (74)$$

The geodesic equations are then

$$\dot{t} = \frac{(r^3 + a^2r + 2Ma^2)E - 2aMl}{r\Delta} \quad (75)$$

$$\dot{\phi} = \frac{(r - 2M)l + 2aME}{r\Delta} \quad (76)$$

$$\dot{r}^2 = \frac{1}{3} \tilde{V}(E, l, r) \quad (77)$$

where $E = -p_t$, $l = p_\phi$ and

$$\tilde{V} = E^2(r^3 + a^2r + 2Ma^2) - 2aMEl - (r - 2M)l^2 - m^2r\Delta \quad (78)$$

Circular orbits are obtained with the conditions $\tilde{V} = 0$ and $\partial_r \tilde{V} = 0$, which lead to

$$\tilde{E} = \frac{r^2 - 2Mr \pm a\sqrt{Mr}}{r(r^2 - 3Mr \pm 2a\sqrt{Mr})^{1/2}} \quad (79)$$

$$\tilde{l} = \pm \frac{\sqrt{Mr}(r^2 \mp 2a\sqrt{Mr} + a^2)}{r(r^2 - 3Mr \pm 2a\sqrt{Mr})^{1/2}} \quad (80)$$

where $+$ describes corotating particles and $-$ counterrotating ones.

At this point it is important to underline some features of this results. Circular orbits exist from $r = \infty$ to the limiting photon orbit where $E \to \infty$

$$r_{ph} = 2M \left(1 + \cos\left(\frac{2}{3} \arccos\left(\pm \frac{a}{M}\right)\right)\right) = 2M \left(1 + \cos\left(\frac{2\pi}{3}\right)\right) = 3M \quad \text{q.e.d.}$$

with $a = 0$. Furthermore for $r > r_{ph}$ the circular orbit can be further classified in analogy with the Schwarzschild case. In particular, for $r > r_{ph}$ the circular orbits are

- Bound for $r > r_{mb}$
  - stable: $\partial_r^2 \tilde{V} \leq 0$
  - unstable: $\partial_r^2 \tilde{V} > 0$

- Unbound for $r < r_{mb}$

where the conditions $\tilde{V}(r_{mb}) = \tilde{E} = 1$ and $\partial_r \tilde{V}(r_{mb}) = 0$ define

$$r_{mb} = 2M \mp a + 2\sqrt{M(M \mp a)}^{1/2}$$

$$= \begin{cases} 
5.8M & a = -M \\
4M & a = 0 \\
M & a = M 
\end{cases}$$
Particularly interesting are of course the **stable** circular orbits, which are obtained for $\partial^2 V / \partial r^2 \leq 0$ which implies

$$\left(1 - \tilde{E}^2 \right) \leq \frac{2M}{3r}$$

After some algebra (exercise) it is possible to show

$$r_{ms} = M(3 + Z_2 \pm [(3 + Z_1)(3 + Z_1 + 2Z_2)]^{1/2}) = \left\{ \begin{array}{cl} 9M & a = -M \\ 6M & a = 0 \\ M & a = M \end{array} \right.$$  \hspace{1cm}  (81)

with

$$Z_1 = 1 + (1 - a^2/M^2)^{1/3} \left[ (1 + a/M)^{1/3} + (1 - a/M)^{1/3} \right]$$

$$Z_2 = (3a^2/M^2 + Z_1^2)^{1/2}$$

In analogy with the Schwarzschild case we can compute the binding energy liberated by a particle at rest at infinity and moving down to the marginally stable circular orbit

$$1 - \tilde{E}_{r=M,a=M} = 1 - \frac{1}{\sqrt{3}} \approx 42\%$$

This is an enormous efficiency when compared to other physical processes.

### 3.3.2 Massless particles

Repeating for Kerr the same analysis carried out for Schwarzschild, we find that

$$\left( \frac{dr}{d\lambda} \right)^{1/2} = \left[ \frac{(r^2 + a^2) - a^2 \Delta}{r^4} \right] (E - V_+)(E - V_-)$$  \hspace{1cm}  (82)

where

$$V_{\pm} = [\omega \pm (\omega^2 - g^{\phi \phi} / g^{tt})^{1/2}]l = \left( \frac{2Mr \pm r^2 \Delta^{1/2}}{(r^2 + a^2) - a^2 \Delta} \right) l$$  \hspace{1cm}  (83)

The coefficient in square brackets in equation (82) is positive and so the photon orbits are allowed for energies such that

$$(E - V_+)(E - V_-) > 0$$  \hspace{1cm}  (84)
Let’s first consider the case of corotating massless particles, i.e. $a > 0$.

Note

- $V_+ > 0 \forall r$
- $V_+(r_+) = 2\omega(r_+)l$ since $g^{\phi\phi}/g^{tt} = 0$ (exercise)
- $V_+(r_0) = 0$
- $V_+ = V_-$ at $r_+$

Since the conditions expressed in (84) must be met, photons must have

(a) $E > V_+$
(b) $E < V_-$

i.e. outside the dashed area. In case (a) these is nothing qualitatively new with respect to the Schwarzschild dynamics. Case (b) instead is interesting as it seems to allow for particle/photon motion having $E < 0$. What does it means? We need a small digression.

### 3.3.3 Penrose process

Let’s consider first the motion of freely falling particles. We know that this motion takes place along timelike geodesics. However, the motion can take place in either direction, i.e. forward or backward in time. Let the particle have four-momentum $\pm p$ and choose an observer with four-velocity $U$. The energy of the particle measured by this observer will be the projection of $p$ along $U$, i.e.

$$ E = -U \cdot p = \begin{cases} > 0 & \text{for } +p \\ < 0 & \text{for } -p \end{cases} $$

In a flat spacetime we conventionally assume particles to move forward in time, so that the measured energy is positive for an observer who is also moving forward in time. Furthermore, if the particle has positive energy relative to one Lorentz observer, it will have positive energy for all observers that move forward in time. This is true in flat spacetime.

In a curved spacetime and in particular in a Kerr spacetime this is no longer true. Consider therefore a representative observer and while all are equivalent, a
well defined one is a Zero Angular Momentum Observer (ZAMO). This observer has zero angular momentum and is at a fixed position in $r$ and $\theta$, but will change its position in $\phi$ as result of frame dragging; these are the equivalent of static observers in a Schwarzschild spacetime. In this case (exercise) the ZAMO 4-velocities take the form

$$U^r_Z = 0; \quad U^\theta_Z = \frac{g_{\theta\phi}}{(g_{\phi\phi})^2 - g_{tt}g_{\phi\phi}} := A > 0; \quad U^\phi_Z = \frac{g_{\theta\phi}}{g_{tt}g_{\phi\phi} - (g_{\theta\phi})^2} = \omega A$$

so that

$$E_Z = -U_Z \cdot p = -(p_tU^t_Z + p_\phi U^\phi_Z) = U^r_Z(-p_t - p_\phi\omega) = A(E - l\omega) \quad (85)$$

Hence, requiring that $E_Z > 0$ it follows

$$\boxed{E > l\omega} \quad (86)$$

Going back to the definition of $V_+$ we can see that all corotating particles with $E > V_+$ will satisfy (86). Similarly, all corotating particles with $E < V_+$ will not. It follows that allowed particles (i.e. with measured positive energy) must have $E > V_+$.

We can repeat the analysis for counterrotating particles, i.e. $al < 0$, and we would find that also in this case the condition for positive energy measured by a ZAMO is $E > V_+$. However, in this case the potentials flip over and so $E > V_+$ also admits particles with energies $E < 0$. This happens for particles in the region $r_+ < r \leq r_0$. In this region a ZAMO would measure as having positive energy a particle which has negative energy at infinity.

Take a particle with energy $E = 0$ that is counterrotating and split it in two inside the ergosphere ($r_+ < r \leq r_0$).
The resulting particle will have the same total energy \( E = E_1 + E_2 = 0 \) so that \( E_1 = -E_2 \). The particle with \( E_2 < 0 \) at infinity will nevertheless have positive energy relative to the ZAMO. The other particle can be directed in such a way that it leaves the ergosphere and reach spatial infinity. The net budget at infinity is: \( E_{\text{initial}} = 0 \) and \( E_{\text{final}} = E_1 > 0 \). Some energy has been extracted from the black hole, which has indeed captured the particle with \( E_2 < 0 \). Note that both particles had positive energy when measured by the ZAMO. There are no contradictions! This is the **Penrose process** and is a generic process to extract energy from a BH even in classical general relativity.

Note that for a particle to escape it must have a suitable negative (it’s counterrotating) angular momentum. In particular

\[
\begin{align*}
E_1 &= E_2 > 0 \\
|l_1| &< |l_2| \quad \text{and} \quad l_2 < 0, l_1 < 0
\end{align*}
\]

In this way the BH acquires a negative angular momentum and slows down!

Recap: In a Penrose process the BH loses both mass and angular momentum.

This is not in contrast with the **Hawking area theorem**, which states that any interaction with a BH leads to an increase of its area. In the case of Schwarzschild is obvious since an increase in mass directly translates into an increase in area. In Kerr

\[
A_{BH} = \int_{r=r_+} \sqrt{\gamma} d\Omega = \int_{r=r_+} 2Mr_+ \sin \theta d\theta d\phi = 8\pi Mr_+
\]

\[= 8\pi M (M + (M^2 - a^2)^{1/2}) \quad (87)\]

with

\[
ds^2 = (r_+^2 + a^2 \cos^2 \theta) d\theta^2 + \frac{(2Mr_+)^2 \sin^2 \theta}{(r_+^2 + a^2 \cos^2 \theta)} d\phi^2
\]

Note: for \( a = 0 \)

\[
8\pi M (M + (M^2 - a^2)^{1/2}) = 4\pi (2M)^2 = 4\pi R_S^2
\]

The Hawking theorem states that \( \delta A_{BH} > 0 \) and the Penrose process requires \( \delta M < 0 \) and yet the two conditions are compatible. In particular it is possible to show (exercise) that for \( a \approx M \)

\[
\frac{M}{a} \left( \frac{\delta M}{\delta a} \right) > 1 \iff \delta A_{BH} > 0
\]
Hence, as long as both $\delta M$ and $\delta a$ are negative (the black holes accretes a particle with negative energy and negative angular momentum) the Penrose process yields an increase in the area.

Let’s ask now a question: If the mass of the BH can be decreased, can it be taken to zero? The answer is no, the Penrose process decrease to zero only the "reducible" mass, leaving behind the "irreducible" mass $M_{irr}$

$$M^2 = M_{irr}^2 + \frac{4\pi J^2}{A_{BH}} = M_{irr}^2 + \frac{J^2}{4M_{irr}^2} \quad (88)$$

where $M_{irr}$ is defined by the Christodoulon formula

$$M_{irr}^2 = \frac{A_{BH}}{16\pi} = \frac{1}{2}(M^2 + \sqrt{M^4 - J^2})$$

and $J = aM$. Clearly if $J \to 0$ then $M \to M_{irr}$ (there is no more energy extracted when the BH has been spin down).

Note:

$$\frac{M_{irr}}{M} = \sqrt{\frac{M_{irr}^2}{M_{irr}^2 + J^2/4M_{irr}^2}}$$

$$= \sqrt{\frac{1}{2}(1 + \sqrt{1 - J^2/M^2})} = \sqrt{\frac{1}{2}(1 + \sqrt{1 - a^2/M^2})}$$

$$= \begin{cases} 1 & a = 0 \\ \left(\frac{1}{2}\right)^{1/2} & a = M \end{cases}$$
Laws of black-hole mechanics / thermodynamics

The horizon has constant surface gravity for a stationary black hole.

Surface gravity is well defined concept in Newtonian gravity but less so in general relativity and in particular in the case of a black hole.

A calculation experienced on the surface of an object of mass $M$

\[ g = \frac{G M}{r^2} = \frac{4}{3} \pi G \rho r \propto r \quad \text{for } \rho = \text{const} \]

\[ M = \rho V = \rho \left( \frac{4}{3} \pi r^3 \right) \]
For a black hole we need to introduce the concept of killing horizon, in this case the surface gravity is the acceleration, as exerted at infinity, necessary to keep an object at the horizon.

\( \xi : \) killing vector

\( \xi^a \nabla_a \xi^b = k \xi^b \)

\( k : \) surface gravity

\( \xi \cdot \xi = -1 \); \( \xi^a \partial_a = \partial_t \) for Schwarzschild

\( \xi^a \partial_a = Q_t + 2Q_\phi \) for Kerr

\( \xi^a \nabla_a \xi^b = \xi^a \nabla^b \xi_a = k \xi^b \)

Tonde Holobongton Finkelstein

\( r \rightarrow r^* = t + r + 2M \ln |r - 2M| \); \( ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{r^2}{1 - \frac{2M}{r}}d\Omega^2 \)

\( \xi^t = (1, 0, 0, 0) \)
\[ \xi^1 = 0, \quad \xi_0 = (1, 0, 0, 0) \]
\[ \xi_{10} = \left( -1 + \frac{2M}{r}, 1, 0, 0 \right) \]
\[ \xi^a \nabla_a \xi^B = k \xi^B \quad \iff \quad -\frac{1}{2} \sigma \left( -1 + \frac{2M}{r} \right) = k \quad \Rightarrow \]
\[ -\xi^a \nabla_a \xi^0 = k \xi^0 \quad -\xi^a \nabla_a \xi^0 = k \xi^0 \quad -\sigma \left( -1 + \frac{2M}{r} \right) = k \]

\textbf{Exercise}

Derive the event horizon of a Kerr black hole

\[ k = \frac{r_+ - r_-}{2(r_+^2 + a^2)} = \frac{\sqrt{M^2 - \frac{a^2}{M^2}}}{2M^2} \]

\[ \sigma = 0 \quad \Rightarrow \quad \frac{M}{4M^2} = \frac{1}{4M} \quad \checkmark \]

\[ \nabla_\xi (\xi_0^2) = 0 \quad \text{Killing eq.} \]
\[ \nabla_\xi \xi = -\nabla_0 \xi_0 \]

(34c)
1st
\[ dM = \frac{K}{8\pi} dA + S d\tau + \phi dQ \]
A: area; \( S \): angular velocity; \( \tau \): ang. mom.
\( \phi \): electrostatic potential; \( Q \): charge

2nd
\[ \frac{dA}{dt} > 0 \quad : \text{the area can only grow} \]

3rd
It is not possible to form a black hole with vanishing surface gravity in a finite time (\( K > 0 \) for \( t < \text{finite} \)).
4 Relativistic stars

4.1 Non-rotating stars

4.1.1 Conservation equations

Let’s go back to the Einstein equations and drop the assumption of vacuum. The RHS is now non-zero and is given by the energy-momentum tensor

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} = 8\pi [(e + p)U_\alpha U_\beta + pg_{\alpha\beta}]$$

(89)

where $e = \rho(1 + \epsilon)$ is the energy density, $\epsilon$ is the specific internal energy, $\rho$ is the rest-mass density, $p$ is the (isotropic) pressure and $U$ is the fluid four-velocity. The presence of a fluid implies two additional tensor equations. The conservation of energy and momentum

$$\nabla_\mu T^{\mu\nu} = 0$$

(90)

and the conservation of the rest-mass

$$\nabla_\mu (\rho U^\mu) = \nabla_\mu J^\mu = 0$$

(91)

Equations (90) and (91) are 4+1=5 equations and represent the equation of relativistic hydrodynamics. In general there are 6 unknowns $\{U^\mu(3), \rho(1), p(1), \epsilon(1)\}$, hence we need an additional equation to chose the system. This is the equation of state (EOS)

$$p = p(\rho, \epsilon, \ldots)$$

The equations of conservation of energy and momentum come after projecting $\nabla \cdot T = 0$ in the direction along the fluid four-velocity (energy equation) and orthogonal to it (momentum equations).

Introducing the specific enthalpy

$$h := \frac{e + p}{\rho} = 1 + \epsilon + \frac{p}{\rho}$$

(92)

it is possible to rewrite the energy-momentum tensor as

$$T_{\mu\nu} = \rho h U_\mu U_\nu + pg_{\mu\nu}$$

(93)

Reminding the definition of the projection tensor

$$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$$

(94)

we can compute the projection of $\nabla \cdot T = 0$ orthogonal to $U$

$$h \cdot (\nabla \cdot T) = 0 \Leftrightarrow h^\nu \nabla_\mu T^{\mu\lambda} = 0$$

(95)

With some algebra

$$h^\nu \nabla_\mu T^{\mu\lambda} = h^\nu \lambda [U_\mu U^\lambda \nabla_\mu (\rho h) + \rho h U_\mu \nabla_\mu U^\lambda + \rho h U^\lambda \nabla_\mu U_\mu + g^{\lambda\mu} \nabla_\nu p]$$

$$= \rho h U_\mu \nabla_\mu U^\nu + h^\nu \lambda g^{\lambda\mu} \nabla_\nu p = 0$$

51
\[ U^\mu \nabla_\mu U^\nu + \frac{h^\nu}{\rho h} \nabla_\mu p = 0 \Leftrightarrow \rho h a^\nu = -(g^{\mu \nu} + U^\mu U^\nu) \nabla_\mu p \quad (96) \]

Clearly if \( p = 0 \) or \( \nabla_\mu p = 0 \), \( a^\nu = 0 \) and the motion is geodetic (this is the case of a collisionless fluid for which \( p = 0 \)). Equations (96) express the conservation of momentum and it’s easy to recognise the well known Euler equations in the Newtonian limit

\[ \partial_t v^i + v^j \partial_j v^i = -\frac{1}{\rho} \partial_t p \]

Similarly

\[ U \cdot (\nabla \cdot \mathbf{T}) = 0 \Leftrightarrow U^\mu \nabla_\mu e + \rho h \nabla_\mu U^\mu \quad (97) \]

Using the continuity equation

\[ \nabla_\mu (\rho U^\mu) = 0 \Leftrightarrow \nabla_\mu U^\mu = -\frac{1}{\rho} U^\mu \nabla_\mu \rho \quad (98) \]

one obtain

\[ U^\mu \nabla_\mu e - h U^\mu \nabla_\mu \rho = 0 \quad (99) \]

to be compared with the Newtonian expression

\[ \partial_t \left( \frac{1}{2} \rho v^2 + \rho e \right) + \nabla \cdot \left( \left[ \frac{1}{2} \rho v^2 + \rho e + p \right] \mathbf{v} \right) = 0 \]

Note that using the first law of thermodynamic

\[ de = hdp + \rho T ds \quad (100) \]

(where \( T \) is the temperature and \( s \) is the specific entropy) equation (99) implies that perfect fluids are adiabatic, i.e.

\[ U^\mu \nabla_\mu s = 0 \quad (101) \]

4.1.2 TOV equations

Let’s consider equation (96) in a spherically symmetric static spacetime with the line element

\[ ds^2 = -e^{2\phi} dt^2 + e^{2\lambda} dr^2 + r^2 d\Omega^2 = -e^{2\phi} dt^2 + \left( 1 - \frac{2M(r)}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 \quad (102) \]

where we have defined

\[ m(r) = \frac{1}{2} r \left( 1 - \frac{1}{g_{rr}} \right) = \frac{1}{2} r (1 - e^{-2\lambda}) \]

and

\[ g_{rr} = \left( 1 - \frac{2M}{r} \right)^{-1} \]
It is then not difficult to show (exercise) that from equation (96) follows
\[
\frac{dp}{dr} = -(e + p) \frac{d\phi}{dr}
\]
(103) to be compared with the equation of Newtonian hydrostatic equilibrium
\[
\frac{dp}{d\rho} = -\rho \frac{d\Phi}{dr}
\]
Similarly, the Einstein equations yield
\[
\frac{d}{dr} m(r) = 4\pi r^2 e
\]
(104)
\[
\frac{dp}{dr} = -(e + p) \frac{d\phi}{dr} = -\frac{(e + p)(m + 4\pi r^3 p)}{r(r - 2m)}
\]
(105)
which are the Tolman-Oppenheimer-Volkov (TOV) equations. \(m(r)\) gives the mass-energy of the system within the radius \(r\).

### 4.1.3 Gravitational mass and density profile

The **gravitational mass** is given by
\[
M = \int_0^{\infty} m(r)dr = \int_0^{\infty} 4\pi r^2 e dr
\]
(106)
Note that the gravitational mass is the mass whose gravitational effects we can measure, but is not the mass we would compute from summing up the rest masses of all the particles in the star. The latter is called **baryon mass** (or **rest mass**) and is defined as the integral in proper volume of the rest-mass density
\[
M_b = \int \rho W d^3\tilde{x} = \int \rho W \sqrt{\gamma} d^3x
\]
(107)
where \(\int d^3\tilde{x} = \int \sqrt{\text{det}(\gamma_{ij})} d^3x\) is the proper volume (\(\gamma_{ij}\) is the spatial part of the metric), \(d^3\tilde{x}\) is the coordinate volume and \(\rho W\) is the density measured by an observer with \(W\).
Note that because \(M\) measures an energy and includes also the (negative) contributions of the binding energy
\[
M_b > M
\]
(108)
The metric of these spacetime needs to be split in an exterior region and an interior one. The **exterior** is Schwarzschild, since we are in vacuum, while the **interior** one can be computed from the numerical solution of the TOV equations.
A particularly simple solution id the one given by a uniform-density star, i.e.
\[
\bar{e} = \frac{M}{\frac{4}{3}\pi R^3}
\]
In this case the metric is

\[ g_{rr} = e^{-2\Lambda} = \begin{cases} 
1 - \frac{4\pi r^2}{3}r^{-1} & \text{for } r \leq R \\
1 - \frac{2M}{r} & \text{for } r > R 
\end{cases} \]

and

\[ \sqrt{g_{tt}} = \begin{cases} 
\frac{3}{2} \left(1 - \frac{2M}{R}\right)^{1/2} - \frac{1}{2} \left(1 - \frac{2Mr^2}{R^3}\right)^{1/2} & \text{for } r \leq R \\
1 - \frac{2M}{r} & \text{for } r > R 
\end{cases} \]

Note: the mass and the radius are not linearly independent but there is a limit in the compactness \( C = M/R \). This can be appreciated by looking at the pressure profile (which is not constant!)

\[ p(r) = \bar{\epsilon} \left[ \frac{(1 - 2Mr^3/R^3)^{1/2}}{3(1 - 2M/R)^{1/2} - (1 - 2Mr^2/R^3)^{1/2}} \right] \]  \hspace{1cm} (109)

from which the central pressure is

\[ p_c = p(r = 0) = \bar{\epsilon} \left[ \frac{1 - (1 - 2M/R)^{1/2}}{3(1 - 2M/R)^{1/2} - 1} \right] \]

Note that \( p_c \to 0 \) for \( C \to 4/9 \) which can be defined as the critical compactness for a relativistic star. For a stable solution also

\[ C \leq \frac{4}{9} \approx 0.44 \]

or equivalently

\[ R \geq \frac{9}{8} R_S \]

where \( R_S = 2M \). A star must have a radius which is larger of its Schwarzschild radius by at least \( 1/8 \) (12%).
$M_{\text{max}}$ depends on the EOS, which is unknown. We know however that $M_{\text{max}} \geq 2M_{\odot}$ (most neutron stars have masses $\approx 1.3 - 1.4M_{\odot}$).

Let’s now ask the question: what happens for a star as an equilibrium solution of the equations on the unstable branch? Besides equilibrium, it is also possible to study the stability of the solutions by performing perturbative analyses of the equilibrium solutions. We will not have the time to discuss the analyses, but we can look at the results. The condition $\partial M/\partial \rho_c = 0$ for spherical stars marks the limit of stability, i.e. the location at which the real part of the fundamental mode of oscillation tends to zero, signalling the onset of a quasi-radial instability.

Stellar models with $\partial M/\partial \rho_c > 0$ will be stable, while models with $\partial M/\partial \rho_c < 0$ will be dynamically unstable. This is rather simple to understand as in the stable branch an increase in mass is associated by an increase in central density. We can ask ourselves the question of what happens to an equilibrium model that is on the unstable branch. It is surely possible to construct such a model in equilibrium (cfr. pen on its tip). A stellar model has two options: move to larger central densities and eventually produce a black hole or move to smaller central densities and settle on the stable branch with same baryon mass. In this case the star expands like a compressed spring which is free to expand.
Note that the transition to the equilibrium configuration is dynamical and because of this is a fully non-linear system, the configuration B is never reached and the star will oscillate around it. If dissipative processes are present, the oscillation will damp out and the system will reach an equilibrium stable configuration with the same baryonic mass but different gravitational masses. The difference in gravitational mass will be the consequence of the fact that the energy-density distribution will have changed (radius is different) and hence its integral

\[ M = \int 4\pi r^2 \rho\, dr \]

4.2 Rotating stars

4.2.1 Slow rotation approximation

Consider a stationary, axially symmetric spacetime, whose line element reads

\[ ds^2 = -H^2(r, \theta) dt^2 + Q^2(r, \theta) dr^2 + r^2 K^2(r, \theta) [d\theta^2 + \sin^2 \theta (d\phi + L(r, \theta) dt)^2] \]

As for Kerr, let \( \Omega = u^\phi / u^t \), then the metric functions \( H, Q \) and \( K \) can only depend on even powers of \( \Omega \), while \( L \) can depend on odd powers of \( \Omega \) (the system must behave to a sign change in \( \Omega \) as to a sign change in \( t \)). As a result, if we consider only first-order corrections in \( \Omega \), the only change in the metric appears in the function \( L(r, \theta) \) and the other functions maintain the same spherical-symmetry expressions, i.e. \( H^2 = e^{2\phi}, Q^2 = e^{2\lambda} \) and \( K^2 = 1 \), so that

\[ ds^2 = -e^{2\phi} dt^2 + e^{2\lambda} dr^2 + r^2 [d\theta^2 + \sin^2 \theta (d\phi - \omega dt - \omega dt)^2] + O(\Omega^3) \]

where \( L(r, \theta) = \omega(r, \theta) + O(\Omega^3) \). As in Kerr \( \omega(r, \theta) \) is the frame-dragging angular velocity, so that

\[ \tilde{\omega} = \Omega - \omega[r, \theta] \]

is the velocity of a fluid element relative to a zero-angular-momentum observer. The solution of the equilibrium equations in the slow-rotation approximation yields the same TOV equations encountered for spherically symmetric systems with the addition of the "\( t\phi \)" component of the Einstein equations.

\[ \frac{dm(r)}{dr} = 4\pi r^2 e \]
\[ \frac{dp}{dr} = -(e + p) \frac{d\phi}{dr} = -\frac{(e + p)(m + 4\pi r^3 p)}{r(r - 2m)} \]
\[ \frac{1}{r^4} \frac{d}{dr} \left( r^4 j \frac{d\omega}{dr} \right) + 4\tilde{\omega} \frac{dj}{r \, dr} = 0 \]

where \( e^{-2\lambda} = 1 - 2m(r)/r \) and \( j(r) = e^{-(\nu + \lambda)} \).

Note:

- Because the star is (slowly) rotating, the spacetime has non-zero angular momentum 4-vector \( J \)

\[ \mathbf{J} := \mathbf{T} \cdot \xi = T^\mu_\rho \xi^\rho \]

56
where $T$ is the energy-momentum tensor and $\xi_\phi^\mu$ is the axial Killing vector, so that

$$J^\mu = T^\mu_\phi$$

The angular momentum is then given by

$$J := \int_{t=\text{const}} \sqrt{-g} J^\mu d^3 x = \int_{t=\text{const}} T^t_\phi \sqrt{-g} d^3 x$$

which allows for the definition of the momentum of inertia

$$I := \frac{J}{\Omega}$$

(115)

Note that $I \rightarrow I(0)$ for $\Omega \rightarrow 0$, i.e. it doesn’t diverge.

- Outside the star ($r > R$) the metric has the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left[d\theta^2 + \sin^2 \theta \left(d\phi - \frac{2J}{r^3} dt\right)\right]^2$$

and

$$\tilde{\omega} = \Omega - \frac{2J}{r^3}$$

where $\Omega$ and $J$ are constant. Taking the derivative

$$J = \frac{1}{6} r^4 \frac{d\tilde{\omega}}{dr}$$

and so $J$ is read from the solution of $\tilde{\omega}$ at $r = R$.

- In the slow-rotation approximation the star still retains a spherical slope, despite having angular momentum. Expansions at second order of $\Omega$ will introduce changes in the shape and a quadrupole moment $Q$. We recall that for a Kerr BH there is a very simple relation that links all the higher mass multipoles $M_l$ and mass-current multipoles $S_l$ to the basic parameters of the BH, i.e. mass $M$ and angular momentum $J$

$$M_l + iS_l = M \left(\frac{iJ}{M}\right)^l$$

(116)

which is the so-called no-hair theorem.

For $l = 1$ we obtain $M_1 = 0$ and $S_1 = J$ which means that the 1st multipole is the angular momentum. For $l = 2$ we obtain the mass quadrupole since $M_2 = -J^2/M$ and $S_2 = 0$. For $l = 3$ we have the current octupole with $M_3 = 0$ and $S_3 = -J^3/M^2$ and, concluding, for $l = 4$ we have the mass hexapole with $M_4 = -J^4/M^3$ and $S_4 = 0$.

- The moment of inertia inside the star can be computed as

$$I = \frac{2}{3} \int_0^R r^3 \frac{dj}{dr} \left(\frac{\tilde{\omega}}{\Omega}\right) dr$$

using the additional Einstein equation.

- The new equilibria can be computed along sequences of constant angular momentum and yield configurations with larger or smaller $\rho_c$. 

57
4.2.2 Mass-shedding limit

If the star is spinning at very high rates, perturbative approaches are no longer adequate/convenient. Best obtain solutions via numerical integration of Einstein equations in two dimensions \((r, \theta)\).

As the star is spun up, the shape will change, the quadrupole moment increase up until a cusp appears at the equator. A fluid element at the cups is moving at the Keplerian frequency and hence an a geodesic. This is the mass-shedding limit: the star loses mass at higher rotation rates.

For a given central density it is possible to find a value of the angular momentum corresponding to a mass-shedding configuration. These configurations will all have the property of \(\Omega = \Omega_K\), although each configuration will have a different \(\Omega_K\) and of course \(J\).

The end result will be a region in the \((M, \rho_c)\) plane comprising all equilibrium models for uniformly rotating stars.
A necessary analytical stability criterion is not known for rotating stars. A simple criterion is that of the turning point

\[
\frac{\partial M}{\partial \rho_c} \bigg|_{J = \text{const}} > 0
\]

so that the maximum of curves of constant angular momentum can be taken to mark the maximum mass for that sequence.

This is a reasonable first approximation as shown via numerical simulations. Let’s now ask the following question: What happens to a star that is born near the mass-shedding limit and spins down due to some energy loss while conserving the baryon mass?

Stars A and B will spin down to two stable non-rotating configurations A' and B'. Star C will collapse to produce a rotating black hole.
4.3 Collapse of a dust sphere to a black hole

So far we have concentrated on stationary configurations but the gravitational collapse is clearly a dynamical process involving considerable portions of spacetime. Also in this case, it is useful to start studying a simplified scenario as the one offered by the collapse of a star composed of uniform-density pressureless dust. This is also known as the Oppenheimer-Snyder (OS) collapse. In this case, in fact, the fluid motion is particularly simple (i.e. it is that of collisionless particles having a collective motion in the same direction) and the spherical symmetry (via the Birkhoff’s theorem) guarantees that the only portion of spacetime that is undergoing an effective evolution is the stellar interior one, since the exterior always remain that of a Schwarzschild solution (albeit with dynamical boundary).

Before looking at the details of the dynamics it is useful to consider the set of equations, both Einstein and hydrodynamical, that describe the process; as we will see, these equations are well known also in different (cosmological) context. We start considering a spherically symmetric diagonal line element of the form

\[ ds^2 = -a^2 dt^2 + b^2 dr^2 + R^2 d\Omega^2 \quad (117) \]

where \( a \) and \( b \) are functions of \((r, \theta)\). Here, \( R \) is a circumferential radial coordinate since the proper circumference is calculated simply as

\[ C = \int_{r, \theta = \text{const}} \sqrt{ds^2} = \int \sqrt{g_{\phi \phi}} d\phi = 2\pi R \quad (118) \]

Adopting a set of comoving coordinates, the four-velocity is \( u^\alpha = (u^0, 0, 0, 0) \), and since \( u^\alpha u_\alpha = -1 \), we have that

\[ u^\alpha = (a^{-1}, 0, 0, 0) \quad u_\alpha = (-a, 0, 0, 0) \quad (119) \]

To cast the hydrodynamic equations in a form that resembles the Newtonian one, it is better to introduce differential operators that measure variations with respect to the proper "distance". In general

\[ \frac{\partial}{\partial (\text{proper } x^3 \text{ coordinate})} = \frac{\partial}{\sqrt{g_{\alpha\beta}} \partial x^3} \quad (120) \]

As a result we can introduce the differential operators

\[ D_t = \text{proper time derivative} = \frac{1}{a} \partial_t \]

\[ D_r = \text{proper radius derivative} = \frac{1}{b} \partial_r \quad (122) \]

such that

\[ u = D_t R = \frac{1}{a} \partial_t R, \quad \Gamma = D_r R = \frac{1}{a} \partial_r R \quad (123) \]

so that \( u \) is the radial component of a four-velocity in a coordinate system that has \( R \) as the radial coordinate, while \( \Gamma \) measures the variation of the circumferential radius with respect to the radial coordinate (\( \Gamma = 1 \) is a flat spacetime). Within this framework, the full set of hydrodynamics and field equations is given by
• Conservation of energy
\[
\frac{D_t e}{e + p} = \frac{D_t \rho}{\rho}
\] (124)

• Conservation of baryon number
\[
\frac{D_t \rho}{\rho} = \frac{1}{R^2} \partial_r (u R^2)
\] (125)

• Conservation of momentum
\[
D_t u = - \frac{\Gamma}{e + p} D_r p - \frac{m}{R^2} - 4\pi p R
\] (126)

• Einstein equations
\[
D_t \Gamma = - \frac{u}{e + p} D_r p
\] (127)
\[
D_t m = -4\pi R^2 u e
\] (128)
\[
\Gamma = 1 + u^2 - \frac{2m}{R}
\] (129)

Note that equation (129) indicates how \( \Gamma \) is the general relativistic analogue of the Lorentz factor in special relativity (\( \Gamma = 1 \) in Newtonian physics). Equations (124)-(129), together with an equation of state, represent the set of equations to be solved to compute the evolution of the interior spacetime of a spherical collapse.

Note also that since the dust particles will be collisionless and all having the same radial motion, \( p = 0 \) and this simplifies the above set of equations considerably. Also, since the rest-mass is conserved during the collapse we can introduce a new variable that labels different shells with the rest-mass they contain, i.e.
\[
\mu(r) = \int 4\pi R^2 \rho b dr
\] (130)

where \( \rho \) is the rest-mass density. Clearly, this parametrisation is valid as long as each shell does not interact with the neighbouring ones, i.e. there is no "shell crossing".

Let us now consider in detail the consequences of the assumption that the fluid is homogeneous, i.e. \( D_r p = 0 = D_r \rho \). In this case, equation (127) reduces to \( D_t \Gamma = 0 \), so that \( \Gamma = \Gamma(\mu) \) only and
\[
m = \int_0^{R_0} 4\pi R^2 e dR = \frac{4\pi}{3} R_0^3 e
\] (131)

Let us now adopt a "comoving-observer gouge", i.e. a gauge in which the time coordinate is the proper time on a line \( dx^i = 0 \) and such that \( g_{00} = a = 1 \) or, equivalently, \( D_t = \partial_\tau \). Furthermore, because of the homogeneity at any point, we can decompose \( R = R(\mu, t) \) as \( R = \tilde{F}(t) \tilde{R}(\mu) \), so that
\[
\dot{\tilde{R}} = \partial_\tau \tilde{R} = u = \tilde{F} \tilde{R} = \frac{\dot{\tilde{F}}}{\tilde{F}} \tilde{R}
\] (132)
and the Einstein equation (129) becomes

\[ \Gamma^2 = 1 + \frac{a^2}{R} = 1 + R^2 \left[ \left( \frac{F'}{F} \right)^2 - \frac{8\pi e}{3} \right] = 1 - k \frac{R^2(\mu, t)}{S^2(t)} \]  \hspace{1cm} (133)

where \( k = 0, \pm 1 \) accounts for the sign of the term in square brackets and \( S \) is a function of time only and just a shorthand for what is contained in the square brackets. Because of the decomposition of \( R \), the ratio \( \tilde{R}/S \) is a function of \( r \) only and thus we can simply write

\[ \Gamma^2 = 1 - kr^2 \]  \hspace{1cm} (134)

With \( R = Sr \)

\[ b = \frac{1}{r} \partial_r R = \frac{S}{\Gamma} = \frac{S}{(1 - kr^2)^{1/2}} \Leftrightarrow b^2 dr^2 = \frac{S}{1 - kr^2} dr^2 \]

and \( R^2 d\Omega^2 = S^2 r^2 d\Omega^2 \) the line element (117) becomes

\[ ds^2 = -dt^2 + S^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \]  \hspace{1cm} (135)

Clearly, this line element is the metric of a Friedmann-Robertson-Walker (FRW) cosmological solution

\[ ds^2 = -dt^2 + S^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) = -dt^2 + S^2(t) (d\chi^2 + f^2(\chi) d\Omega^2) \]

with

\[ \chi = \begin{cases} \sinh^{-1} r & \text{for } k = -1 \\ r & \text{for } k = 0 \\ \sin^{-1} r & \text{for } k = 1 \end{cases} \quad \quad f(\chi) = \begin{cases} \sinh r & \text{for } k = -1 \\ r & \text{for } k = 0 \\ \sin r & \text{for } k = 1 \end{cases} \]

where the function \( S \) (i.e. the conformal factor of the spatial part of the metric) is simply the "scale factor". Similarly, it will not be surprising that when expressed in this metric, the hydrodynamic and Einstein equations will essentially reduce to the Friedmann equations

\( \ddot{S} = -\frac{4\pi}{3} (p + e) S \)  \hspace{1cm} (136)

\( \dot{S} = \frac{8\pi}{3} c S^2 - k \)  \hspace{1cm} (137)

Started differently, the spatial part of the line element (135) describes geometries with different constant curvatures (i.e. the curvature is the same everywhere but is not constant in time), with the different geometries being selected by the values of the coefficient \( k \).

In other words, in spherical symmetry, the dynamical spacetime of a collapsing (expanding) region occupied by homogeneous matter is a FRW-universe. Cosmologically, there are three possible solutions according to the value of \( k \) and thus on the constant curvature \( (k = -1, \text{ curved open universe}; k = 0, \text{ flat universe}; k = 1, \text{ curved closed universe}) \). Clearly, the relevant solution
in the context of an OS collapse is the one with positive constant curvature ($k = 1$) in which case the line element can be expressed in terms of comoving hyperspherical coordinates ($\chi, \theta, \phi$)

$$ds^2 = -d\tau^2 + S^2(\tau)(d\chi^2 + \sin^2\chi d\Omega^2)$$  \hfill (138)

where $\chi = \arcsin r$.

There is an important difference between the FRW universe and the spacetime of an OS collapse since in the latter case not all the spacetime is occupied by matter (the dust sphere has initially a finite radial size $R_0$) and the vacuum portion (i.e. for $R > R_0$) will be described by a Schwarzschild spacetime. The matching between the two portions can be done at the surface of a star by requiring the continuity of the proper circumference

$$C_{Schw} = \int \sqrt{g_{\phi\phi}} d\phi = 2\pi R_0 = C_{FRW} = 2\pi S \sin \chi_0$$ \hfill (139)

Since it must hold at all times, we have that

$$R_0 = S \sin \chi_0$$ \hfill (140)

Let us now consider the equation of motion in the collapsing portion of the spacetime. In this case, equation (128) reduces to $D_t m = -4\pi R^2 u p = 0$, thus implying that $m$ is not a function of time but of radius only, i.e. $m = m(\mu)$, as it should be in the absence of shocks. Similarly, equation (126) reduces to

$$D_t u = -\frac{m}{R^2}$$ \hfill (141)

which is essentially the geodesic equation. The trajectory of any shell can therefore be obtained through a time integration and is given by

$$\dot{R} = \frac{dR}{d\tau} = D_t R = \left(\frac{2m}{R} - \frac{2m}{R_0}\right)^{1/2}$$ \hfill (142)

In other words, as a shell of dust will go from $R_0$ to $R = 0$ in a finite proper time

$$\tau = \frac{\pi}{2} R_0 \left(\frac{R_0}{2M}\right)^{1/2}$$ \hfill (143)

Note that this time is the same for all initial radial positions $R$; this is a trivial consequence of the uniformity in density, for which the ratio $R^3/m(R) = const.$
Note that all worldlines of different radial shells in an OS collapse reach the singularity at the same proper time expressed in equation (143). It is not surprising that this time also corresponds to what we have seen to be the proper time of a radially infalling particle on a Schwarzschild BH.

Once expressed in the coordinate system (138) and after introducing the "cycloid parameter" $\eta \in [0, \pi]$, the equations of motion take the simpler form

$$R = \frac{R_0}{2} (1 + \cos \eta), \quad S(\eta(\tau)) = \frac{S_m}{s} (1 + \cos \eta), \quad \tau = \frac{S_m}{2} (\eta + \sin \eta)$$  \hspace{1cm} (144)

where $\eta$ is essential playing the role of a time coordinate ($\eta = 0$ at the beginning of the collapse and $\eta = \pi$ at the end) and is defined as $d\eta = d\tau / S$.

Using now these last equations and the condition $R_0 = S \sin \chi_0$ one finds that

$$S_m = \left( \frac{R_0^3}{2M} \right)^{1/2}, \quad \chi_0 = \arcsin \left( \frac{2M}{R_0} \right)^{1/2}$$  \hspace{1cm} (145)

Particularly interesting is to calculate the proper time $\tau$ at which a shell initially at $R_0$ reaches $R = 2M$. This can be computed from (144) and is given by

$$\tau_{2M} = \left( \frac{R_0^3}{8M} \right)^{1/2} (\eta_{2M} + \sin \eta_{2M}) = \begin{cases} 0 & \text{for } \eta = 0 \\ \left( \frac{R_0^3}{8M} \right)^{1/2} & \text{for } \eta = \pi \end{cases}$$  \hspace{1cm} (146)

where $\eta_{2M} = \arccos(4M/R_0 - 1)$. The expressions will be useful in the next section when we will discuss what happen to outgoing photons as the collapse proceeds and that may never reach null infinity.

Just to give you an example, for $R_0 = 8M$, $\tau_{2M} = 9.12$ and $\tau(R_0 = 0) = 8\pi = 25$.

### 4.3.1 Trapped surface

Assuming the cosmic censorship to hold, the end-result of the spherical collapse will be a Schwarzschild black hole, the physical singularity will be covered by an event horizon (a null surface that photons cannot leave). However, the Schwarzschild solution will be reached only asymptotically and is interesting to
ask how the event horizon is formed during the collapse; in practice we need to study the trajectory of the \textit{outermost outgoing photon} that was not able to reach null infinity. Similarly, we can conclude where, \textbf{at each instant of the collapse}, the last outgoing photon will be sent and reach null infinity. This surface will mark the outermost trapped surface, i.e. the \textbf{apparent horizon} and by definition will always be \textbf{contained within the event horizon}, i.e. $R_{AH} \leq R_{EH}$.

Let’s us consider therefore the worldline of an outgoing radial photon. In this case, $ds^2 = d\phi = d\theta = 0$ and the line element (138)

$$ds^2 = -d\tau^2 + S^2(\tau)(d\chi^2 + \sin^2 \chi d\Omega^2)$$

then yields the curve

$$\frac{d\chi}{d\tau} = \pm \frac{1}{S(\tau)}$$

(147)

Using now the cycloid parameter $\eta$, it is easy yo show that these photons propagate along straight lines in a $(\chi, \eta)$ plane

$$\frac{d\chi}{d\eta} = \pm 1$$

(148)

or, stated differently, follow curves of the type

$$\chi = \chi_e \pm (\eta - \eta_e)$$

(149)

where $\chi_e$ and $\eta_e$ are the "place" and the "time" of emission, respectively. A swarm of outgoing photons will be trapped if their proper area will not grow in time, i.e. if and only if

$$\frac{dA}{d\eta} \leq 0$$

(150)

where the area $A$ is defined, as usual, as

$$A = \int \sqrt{g_{\phi\phi}g_{\theta\theta}} d\theta d\phi$$

Writing out the condition (150) explicitly yields (exercise)

$$\eta_e \geq \pi - 2\chi_e \Leftrightarrow \chi_e \geq (\pi - \eta_e)/2$$

(151)

In other words, any outgoing photon emitted at $\chi_e$ and time $\eta_e$ will be able to propagate if and only if $\eta_e$ is smaller than $\pi - 2\chi_e$. This will mark a region in the $(\chi, \eta)$ plane, whose border is the trajectory of the marginally trapped photon.
Out of all the possible trapped surfaces, the most important is clearly the outermost one since it will discriminate between the photon that will propagate to null infinity from one that will be trapped. Such a surface at any time selects the apparent horizon (AH), which is a 2-surface, in contrast to the event horizon, which is a 2+1-surface (2 space + 1 time coordinate).

Since the photon must be emitted in the star, $\chi_0 \leq \chi_e$ and so the time of formation of the AH is given by

$$\eta_{AH} = \pi - 2\chi_0 = 2 \arccos \left( \frac{2M}{R_0} \right)^{1/2}$$

(152)

where we used the relations

$$\frac{\pi}{2} = \arcsin x + \arccos x; \quad \chi_0 = \left( \frac{2M}{R_0} \right)^{1/2}$$

Give a collapsing dust cloud of mass $M$ and radius $R$ an AH will form at time $\eta_{AH}$. What is the position of the cloud’s surface at that time? Luckily, answering this question in the case of an OS collapse is particularly simple and reveals that the AH first forms when the stellar surface crosses $R = 2M$. Note: this is true only in OS collapse (exercise).

Finally, we consider the evolution of the event horizon (EH) which is defined as the surface for which the equality condition (150) holds. Using the constraint that the event horizon is always outside or coincides with the apparent horizon, we can set $\chi_{EH} = \chi_{AH}$ when $\eta = \eta_{AH}$, the worldline for the EH is given by

$$\chi_{EH} = \chi_0 + (\eta - \eta_{AH}) \Leftrightarrow \eta = \eta_{AH} + (\chi_{EH} - \chi_0)$$

(153)

for $\eta \leq \eta_{AH}$ (note that $\chi_{EH} = 0$ for $\eta = \chi_0 - \epsilon a_{AH}$). Using now the circumferential radial coordinate we can write that

$$R_{EH} = \frac{1}{2} \left( \frac{R_0^3}{2M} \right)^{1/2} (1 + \cos \eta) \sin(\chi_0 + \eta - \eta_{AH})$$

(154)

An important property to deduce from this last equation is that the EH starts from a zero radius and then progressively grows to reach $R = 2M$; this is to be contrasted with what happens for the AH that is first formed with a non-zero radial size. (Note that the EH grows from zero size will before the AH formed.)
Much of what learnt about the dynamics of trapped surfaces in the OS collapse continues to hold also in the case of the collapse of a perfect fluid, which offers two main differences with respect to the case of dust. The first difference is present already in spherical symmetry and is that the AH is not produced at the time that the stellar surface reaches $R = 2M$ but can, because of the fluid compression, be formed also earlier.
5 Gravitational waves from perturbed black holes

(Most of the material presented here was offered in a summer school at the ICTP in Trieste in July 2002. A more complete discussion of gravitational waves from perturbed black holes and relativistic stars can be found in this article on the archive http://arxiv.org/abs/gr-qc/0302025, which has been published in this book: ICTP Lecture Series, Vol. 3, (2003) ISBN 92-95003-05-5.)

What follows is dedicated to the analysis of the perturbations that characterize a black hole and more precisely a non-rotating (Schwarzschild) black hole. Before discussing in detail black hole perturbations, one might wonder why black hole perturbations are interesting at all. Indeed, there are a number of good reasons why it is interesting and important to consider black hole perturbations. Firstly, the presence of perturbations can break the static properties of a black hole spacetime and be therefore the origin of gravitational waves emission. Secondly, the gravitational waves emitted by a black hole carry information about its properties such as mass, spin and charge. Finally, by investigating the response of black holes to perturbations it is possible to deduce important conclusions on the stability of these objects (being a solution of the Einstein equations, in fact, is just a sufficient condition for stability). Because this is such an important area of research, some of the main results date back to the first studies made by Regge and Wheeler and the subsequent developments that took place in the 70’s.

5.1 Linear perturbation of black holes

The starting point in the analysis of black hole perturbations is, of course, the unperturbed solution represented by a Schwarzschild black hole line element

\[ ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = -(1 - \frac{2M}{r}) dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\Omega^2 \]

where \( \hat{g}_{\mu\nu} \) represents the metric tensor of the static background spacetime. Because the latter is assumed to be vacuum, the Einstein equations assume a more compact form and can be written as

\[ \hat{R}_{\mu\nu} = 0 \]  

where \( \hat{R}_{\mu\nu} \) is the Ricci tensor built with the background metric \( \hat{g}_{\mu\nu} \). If small perturbations \( h_{\mu\nu} \) are now introduced, the resulting metric will be

\[ g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu} \]

where, again, the perturbations considered are much smaller than the background, i.e. \( |h_{\mu\nu}|/|\hat{g}_{\mu\nu}| \ll 1 \).

Just as for a static background, the behaviour of the perturbed spacetime will be expressed by the Einstein equations that, using the notation as in (155) can be written as

\[ R_{\mu\nu} = 0 \]  

where \( R_{\mu\nu} = R_{\mu\nu}(g_{\mu\nu}) \). At first order of approximation, the linearity can be exploited to break up the Einstein equations as

\[ \hat{R}_{\mu\nu} + \delta R_{\mu\nu} = 0 \]  

where \( \delta R_{\mu\nu} = \delta R_{\mu\nu}(g_{\mu\nu}) \). At first order of approximation, the linearity can be exploited to break up the Einstein equations as
where $\delta R_{\mu\nu} = R_{\mu\nu}(h_{\mu\nu})$. Using now equation (155), the field equations reduce to

$$\delta R_{\mu\nu} = 0 \quad (159)$$

A different way of writing this last equations is in terms of the Christoffel symbols and more precisely as

$$\delta R_{\mu\nu} = -\nabla_\beta \delta \Gamma^\beta_{\mu\nu} + \nabla_\nu \delta \Gamma^\beta_{\mu\beta} = 0 \quad (160)$$

where the perturbed Christoffel symbols are defined as

$$\delta \Gamma^\beta_{\mu\nu} = \frac{1}{2} \tilde{g}^{\alpha\beta}(h_{\mu\alpha,\nu} + h_{\nu\alpha,\mu} - h_{\mu\nu,\alpha}) \quad (161)$$

Before proceeding further we should comment on some of the properties of the metric perturbations $h_{\mu\nu}$. An important constraint is posed by Birkhoff’s theorem, which states that the Schwarzschild solution is the only spherically symmetric, asymptotically flat solution of Einstein equations in vacuum even if the spacetime is not static. As a result, non-rotating black holes can only be perturbed by non-radial perturbations and this forces to consider perturbations which complete angular dependence, i.e. $h_{\mu\nu} = h_{\mu\nu}(t, r, \theta, \phi)$. Handling a generic angular dependence can be complicated, but the mathematical treatment can be simplified if the tensor perturbations $h_{\mu\nu}$ are written in a separable form, i.e. as the product of four parts each being a function of one coordinate only. In the case of a scalar function depending on the spatial coordinates only, it is well-known that this can be done after expanding it in series of spherical harmonic functions

$$f(r, \theta, \phi) = \sum_{l, m} a_{lm}(r)Y_{lm}(\theta, \phi) \quad (164)$$

In a similar way, in the case of a vector, the separability is achieved though an expansion in a series of vector spherical harmonics

$$V^\alpha(r, \theta, \phi) = \sum_{l, m} a_{lm}(r)[Y^B_{lm}(\theta, \phi)]^\alpha + \sum_{l, m} b_{lm}(r)[Y^E_{lm}(\theta, \phi)]^\alpha \quad (165)$$

where $Y^B_{lm}(\theta, \phi)$ and $Y^E_{lm}(\theta, \phi)$ are vector spherical harmonics of magnetic (B) and electric (E) type, respectively. It will not therefore surprise that a series expansion of the type (164) and (165) can be used also for a rank-2 symmetric tensor which can then be expanded in a series of tensor spherical harmonics

$$T_{\mu\nu}(t, r, \theta, \phi) = \sum_{l, m} a_{lm}(t, r)[A^\mu^z_{lm}(\theta, \phi)]_{\mu\nu} + \sum_{l, m} b_{lm}(t, r)[B^\mu^z_{lm}(\theta, \phi)]_{\mu\nu} \quad (166)$$
where attention has been paid to the fact that, in general, a rank-2 symmetric
tensor can be expanded in terms of tensor spherical harmonics that behave
differently under parity transformation, i.e. \((A_{lm}^{ax})_{\mu\nu}\) and \((B_{lm}^{pol})_{\mu\nu}\).

More specifically, if \(P\) is the parity operator, that is an operator producing a
parity transformation on a rank-2 symmetric tensor \(F_{\mu\nu}\)
\[
\mathcal{P}(F_{lm}(\theta,\phi))_{\mu\nu} \rightarrow [\tilde{F}_{lm}(\pi - \theta, \pi + \phi)]_{\mu\nu}.
\]

the tensor spherical harmonics can then be classified according to their be-
haviour "under parity change". In practice, are referred as odd or axial (or
sometimes toroidal) those tensor harmonics for which \(\mathcal{P}(F_{\mu\nu}) = (-1)^{l+1}F_{\mu\nu}\).
Similarly, are referred to as even or polar (or sometimes spheroidal) those ten-
sor harmonics for which \(\mathcal{P}(F_{\mu\nu}) = (-1)^{l}F_{\mu\nu}\).
The classification of the tensor spherical harmonics is reflected also on the met-
tric perturbations that, as a result, are classified as "odd" and "even-parity"
respectively. However, before discussing the specific forms that the perturba-
tions assume in these cases, it is convenient to express the metric perturbations
terms of a purely time part \((h_{00})\), of a purely spatial part \((h_{ij})\), and of a mixed
time-spatial part \((h_{0i})\)
\[
h_{\mu\nu} = \begin{pmatrix}
\ h_{00} & h_{i0} \\
\ h_{0i} & h_{ij} 
\end{pmatrix}
\]

In what follows we discuss the basic expressions of \(h_{00}, h_{0i}\) and \(h_{ij}\) for the two
classes of perturbations as well as the equations they satisfy.

5.2 Odd-parity perturbations: the Regge-Wheeler equa-
tion

I first consider the parts of the metric that are of "odd-parity", i.e. the first
term given in the decomposition (166). In this case, it is customary to introduce
the unknown functions \(h_0(t,r), h_1(t,r)\) and \(h_1(t,r)\) so that the components of
(168) can be written as
\[
h_{00} = 0, \\
h_{0i} = h_0(t,r) \left( 0, -\frac{1}{\sin \theta} \sum_{l,m} \partial_\phi Y_{lm}, \sin \theta \sum_{l,m} \partial_\theta Y_{lm} \right) \\
H_{ij} = h_1(t,r)(\hat{e}_1)_{ij} + h_2(t,r)(\hat{e}_2)_{ij}
\]

where \((\hat{e}_{1,2})_{ij} = \sum_{l,m}[(\hat{e}_{1,2})_{ij}]_{lm}\) and we will hereafter omit the \(l, m\) indices
and the sum over them to maintain the expression compact. The tensor spherical
harmonics \((\hat{e}_{1,2})_{ij}\) in (169)-(171) have rather lengthy but otherwise straightfor-
ward expressions which are given by
\[
(\hat{e}_1)_{ij} = \begin{pmatrix}
0 & -\frac{1}{\sin \theta} \partial_\phi Y_{lm} & \sin \theta \partial_\theta Y_{lm} \\
-\frac{1}{\sin \theta} \frac{1}{\sin \theta} \partial_\phi Y_{lm} & 0 & 0 \\
\frac{1}{\sin \theta} \frac{1}{\sin \theta} \partial_\theta Y_{lm} & 0 & 0
\end{pmatrix}
\]
and

\[
(\dot{e}_1)_{ij} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sin \theta} (\frac{\partial^2}{\partial \theta^2} - \cot \theta \partial_{\phi}) Y_{lm} & \frac{1}{2} \left( \frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} - \cos \theta \partial_{\theta} - \sin \theta \partial^2_{\theta} \right) Y_{lm} \right) \\
0 & \frac{1}{2} \left( \frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} - \cos \theta \partial_{\theta} - \sin \theta \partial^2_{\theta} \right) Y_{lm} \right) & \frac{1}{2} \left( \frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} - \cos \theta \partial_{\theta} - \sin \theta \partial^2_{\theta} \right) Y_{lm} \right) & \left( \sin \theta \partial_{\phi} - \cot \theta \partial_{\theta} \right) Y_{lm}
\end{pmatrix}
\]

The Einstein equations with this metric perturbations can be simplified if suitable gauge conditions are chosen. We remind, in fact, that because of the linearized approach, any infinitesimal coordinate transformation will lead to new metric perturbations that are determined after the specification of the suitable conditions for the displacement four-vector \( \xi^\mu \). While these conditions are totally arbitrary, it is convenient to choose those producing a simplification of the equations and, in the case of odd perturbation, the choice usually made is that

\[
h_2(t, r) = 0
\]

In this gauge, which is usually referred to as the "Regge-Wheeler" gauge, the odd-parity metric perturbations assume the simplified form

\[
h^{\alpha\beta}_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & h_0 \\
0 & 0 & 0 & h_1 \\
0 & 0 & 0 & 0 \\
h_0 & h_1 & 0 & 0
\end{pmatrix} \sin \theta \partial_{\theta} P_l (\cos \theta) e^{im\phi}
\]

where \( P_l (\cos \theta) \) are the Legendre polynomial of order \( l \). Besides being simpler, the Einstein equations in this gauge are independent of \( m \) in the sense that the final result will not depend on the specific value chosen for \( m \), which can therefore set to be zero. As a result, the Einstein equations for the perturbed metric (173) lead to the following system of equations

\[
\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r^2} + \left( 1 - \frac{2M}{r} \right) \left[ l(l+1) \frac{r}{r^2} - \frac{6M}{r^3} \right] Q = 0
\]

\[
\frac{\partial h_0}{\partial t} = \frac{\partial}{\partial r} (r_*, Q)
\]

where

\[
Q = \frac{h_1}{r} \left( 1 - \frac{2M}{r} \right)
\]

and

\[
r_* = r + 2M \ln \left( \frac{r}{2M} - 1 \right)
\]

is the "tortoise coordinate". Because \( r_* \rightarrow r \) for \( r \rightarrow \infty \) and \( r_* \rightarrow -\infty \) for \( r \rightarrow 2M^+ \), the tortoise coordinate is particularly suited to study the propagation of perturbations near the black hole event horizon which, in this coordinate system, is placed at \(-\infty\) and therefore does not suffer from coordinate singularities.
A convenient way of looking at the "Regge-Wheeler equation" (174) is that of considering it as a wave equation in a scattering potential barrier $V(r)$, where

$$V(r) = \left(1 - \frac{2M}{r}\right) \left[\frac{l(l+1)}{r^2} - p\right]$$

with $p = \frac{6M}{r^3}$ (178)

This potential is also referred to as the "Regge-Wheeler potential" and has a maximum just outside the event horizon, at $r \approx 3.3M$ in Schwarzschild coordinates.

In this view, the Regge-Wheeler equation shares all of the well-known properties of a wave equation in a scattering potential and the numerous results that have been found for this type of equation (cf. Schrödinger equation) can also be applied to the propagation of perturbations in the spacetime of Schwarzschild black hole.

As an example, a metric perturbation reaching the black hole from spatial infinity can be regarded as a wave packet that will scatter against the potential barrier $V$. As in quantum mechanics, not all of the wave packets will be transmitted through the potential and some of it, depending on the properties of the packet itself, will be reflected and reach again spatial infinity. This is different from what happens, for instance, with a spherical massive shell falling radially onto the black hole. These radical differences underline the importance of a perturbative analysis of black holes spacetimes.

Another interesting aspect of the Regge-Wheeler equation is that it holds in a similar form also for scalar and vector perturbations, with the only difference appearing in the effective potential, where $p = \frac{2M}{r^3}$ for scalar perturbations and $p = 0$ for vector ones.

### 5.3 Even-parity perturbations: the Zerilli equation

Next we consider metric perturbation that are "even-parity" (or polar). The mathematical approach is similar to the one followed for the odd-parity perturbations and also in this case it is useful to introduce a number of unknown
functions $h_0(t,r)$, $h_1(t,r)$, $H_0(t,r)$, $H_1(t,r)$, $H_2(t,r)$, $K(t,r)$ and $G(t,r)$ so that
the perturbed metric functions can be written as
\[
\begin{align*}
    h_{00} &= -\frac{1}{2} \left(1 - \frac{2M}{r}\right)^{1/2} H_0(t,r) Y_{lm} \\
    H_{0i} &= [H_1 Y_{lm}, h_0 \partial_\theta Y_{lm}, h_0 \partial_\phi Y_{lm}] \\
    h_{ij} &= h_1(\hat{f}_1)_{ij} + \frac{H_2}{1 - 2M/r}(\hat{f}_2)_{ij} + r^2 K(\hat{f}_3)_{ij} + r^2 G(\hat{f}_4)_{ij}
\end{align*}
\]
where the tensor spherical harmonics $(\hat{f}_1)_{ij} - (\hat{f}_4)_{ij}$ have the forms
\[
\begin{align*}
    (\hat{f}_1)_{ij} &= \begin{pmatrix} 0 & \partial_\theta Y_{lm} & \partial_\phi Y_{lm} \\ \partial_\theta Y_{lm} & 0 & 0 \\ \partial_\phi Y_{lm} & 0 & 0 \end{pmatrix} \\
    (\hat{f}_2)_{ij} &= \begin{pmatrix} Y_{lm} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
    (\hat{f}_3)_{ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
    (\hat{f}_4)_{ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y_{lm} & 0 \\ 0 & 0 & Y_{lm} \end{pmatrix}
\end{align*}
\]
Also for even-parity perturbations, the gauge freedom can be exploited and in particular a gauge can be chosen in which
\[
G = h_0 = h_1 = 0
\]
As a result of this gauge choice, the polar metric perturbations assume the more compact form
\[
h^{pol}_{\mu\nu} = \begin{pmatrix} H_0(1 - 2M/r) & H_1 & 0 & 0 \\ H_1 & H_2(1 - 2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 K & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta K \end{pmatrix} P_l(\cos \theta)e^{im\phi}
\]
Writing now the Einstein equations for the perturbed metric leads to the following equation
\[
\frac{\partial^2 Z}{\partial t^2} - \frac{\partial^2 Z}{\partial r^2} + \tilde{V} Z = 0
\]
which is also known as the "Zerilli equation". The explicit form of the "Zerilli function" is rather involved but can be expressed, independently on the gauge chosen, as
\[
Z = \frac{4\pi e^{-4\lambda} k_2 + l(l + 1)rk_1}{l(l + 1) - 2 + 6M/r}
\]
where $e^{-\lambda} = 1 - 2M/r$ and the functions $k_1$, $k_2$, $k_3$ and $k_4$ are introduced in place of $G$, $h_1$, $K$, $H_2$ and are defined through the relations
\[
\begin{align*}
    G &= k_3, \quad h_1 = k_4, \quad K = k_1 - \frac{e^{-2\lambda}}{r} \left( r^2 \frac{\partial k_3}{\partial r} - 2k_4 \right), \\
    H_2 &= 2e^{-2\lambda} k_2 + r \frac{\partial k_1}{\partial r} + \left( 1 + r \frac{\partial \lambda}{\partial r} \right) k_1 - e^{-\lambda} \frac{\partial}{\partial r} \left[ r^2 e^{-\lambda} \frac{\partial k_3}{\partial r} - 2e^{-\lambda} k_4 \right]
\end{align*}
\]
Note that as for odd-parity perturbations, the Einstein equations can be recast in the form of a wave equation in a scattering potential barrier $\tilde{V}$, defined as

$$\tilde{V} = \left(1 - \frac{2M}{r}\right) \left[\frac{2q(q + 1)r^3 + 6q^2Mr^2 + 18qM^2r + 18M^3}{r^3(qr + 3M)^3}\right]$$

where $q = (l + 1)(l + 2)/2$.

Interestingly, the Regge-Wheeler and Zerilli equations are closely related and it is possible to transform the first one for axial modes into the second one for polar modes via suitable differential operators.

### 5.4 QNMs of black holes

Since equations (174) and (184) describe the response of the black hole to external perturbations, they are basically telling us about the vibrational modes of such a spacetime. In particular, if a harmonic time dependence is introduced for the perturbation equations (174) and (184), i.e. if $Q, Z \propto e^{i\omega_n t}$ where $\omega_n$ is the oscillation frequency of the $n$-th mode and a complex number of type

$$\omega_n = \omega_{r,n} + i\omega_{i,n}$$

with $n = 1, 2, \ldots$, is then possible to define the Quasi-Normal Modes (QNMs) of the BH as the solution of equations

$$\partial^2_{r^*}Q + [\omega^2 - \tilde{V}]Q = 0$$

$$\partial^2_{r^*}Z + [\omega^2 - \tilde{V}]Z = 0$$

that satisfy a pure outgoing-wave boundary condition at spatial infinity and a pure ingoing-wave boundary condition at the event horizon, i.e. $Q, Z \propto \exp(i\omega r_*)$ for $r_* \to -\infty$ and $Q, Z \propto \exp(-i\omega r_*)$ for $r_* \to \infty$.

The literature on the solution of the Regge-Wheeler and of the Zerilli equations as well as the determination of the perturbation spectrum of BH is vast and there are several reviews on this topic. We therefore briefly summarize what could be considered the main results in the solution of the eigenvalue problem for the QNMs of a Schwarzschild BH:

- All the QNMs of a Schwarzschild BH have positive imaginary parts and represent therefore damped modes. As a result, a Schwarzschild BH is linearly stable against perturbations.

- The damping time of these perturbations depends linearly on the mass of the black hole (i.e. $\omega_n \propto 1/M$) and is shorter for higher-order modes (i.e. $\omega_{i,n+1} > \omega_{i,n}$). As a result, the detection of gravitational waves emitted from a perturbed black hole could provide a direct measurement of its mass.

- The excitation of a black hole and the consequent emission of gravitational radiation is referred to as black hole "ringing". The amplitudes of the gravitational waves emitted during the black hole ringing decay in time and the late-time behaviour (or tail of the ringing) is such that the amplitude evolution can be described in terms of a power law, whose envelope represents the superposition of the various QNMs.
The QNMs in black holes are **isospectral**, i.e. axial and polar perturbations have the same complex eigenfrequencies so that the real and imaginary parts of the spectrum are identical. This is simply due to the uniqueness in which a black hole can react to a perturbation. This is not true for relativistic stars.

The fundamental frequencies of oscillation $\omega_{r,n}/2\pi$ have been computed by a number of authors and are now known up to very large mode numbers. Reported in the table below are the frequencies of the first four modes, together with the values of their decay times $\tau_d$ (i.e. $1/\omega_{i,n}$). The data refers to modes with $l = 1, 2$ and have been computed for $M = M_\odot$.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$n$</th>
<th>$\omega_{r,n}/2\pi$ (kHz)</th>
<th>$\tau_d$ (ms)</th>
<th>$\ell$</th>
<th>$n$</th>
<th>$\omega_{i,n}/2\pi$ (kHz)</th>
<th>$\tau_d$ (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>12.075</td>
<td>5.5344 x 10^{-2}</td>
<td>3</td>
<td>0</td>
<td>19.376</td>
<td>5.3135 x 10^{-3}</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>11.203</td>
<td>1.7983 x 10^{-2}</td>
<td>3</td>
<td>1</td>
<td>18.833</td>
<td>1.7510 x 10^{-3}</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>9.7291</td>
<td>1.0298 x 10^{-2}</td>
<td>3</td>
<td>2</td>
<td>17.834</td>
<td>1.0281 x 10^{-3}</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>8.1294</td>
<td>6.9856 x 10^{-3}</td>
<td>3</td>
<td>3</td>
<td>16.551</td>
<td>7.1354 x 10^{-3}</td>
</tr>
</tbody>
</table>

For any value of the harmonic index $l$, the real part of the frequency $\omega_{r,n}(l)$ approaches a non-zero limiting value as the mode number $n$ increases, while the imaginary part increases linearly $\propto n/4$ (i.e. higher modes have shorter decaying timescales).

Most of what has been presented in this Section for a Schwarzschild black hole can be formulated also for a rotating (Kerr) black hole. In this case, however, the mathematical apparatus is more involved (the potential is, for instance, complex) and some new features, such as the **super radiance** (i.e. the amplified scattering of electromagnetic waves) can take place.
6 3+1 splitting of spacetime

Let $\mathcal{M}$ be a 4-dimensional manifold endowed with a metric $g$; $\mathcal{M}$ can be split in purely spatial hypersurfaces $\Sigma(t)$ ordered by the coordinate $t$.

A and B are spacelike separated.

Given a surface, a first possible operation is the definition of the local normal vector $n$. Defined the one-form (which is telling us how the time is growing) as

$$\Omega_\mu = \nabla_\mu t$$

with the property

$$|\Omega|^2 = \Omega_\mu \Omega^\mu = g^{\mu\nu} \Omega_\mu \Omega_\nu = g^{\mu\nu} (\nabla_\mu t)(\nabla_\nu t) = g^{tt} (\nabla_t t)(\nabla_t t) = g^{tt}$$

the normal vector $n$ to $\Sigma_t$ is

$$n_\mu = A \Omega_\mu$$

where $A = \text{const}$ and with

$$n_\mu n^\mu = A^2 \Omega_\mu \Omega^\mu = A^2 g^{tt} = -1$$

since the normal $n$ should be a timelike unit vector. With the definition $A = \pm \alpha$ follows that $\alpha^2 = -1/g^{tt}$. If we chose the negative sign for $\alpha$ so that the normal vector is directed to the future, we obtain

$$n_\mu = -\alpha \nabla_\mu t$$

or $n_\mu = (\alpha, 0, 0, 0)$, which is the covariant component of the unit normal vector. Note that $\alpha = \alpha(x^\alpha)$, so that it changes in space and time. $\alpha = \alpha(x^\alpha)$ is called lapse function. The contravariant component will be

$$n^\mu = g^{\mu\nu} n_\nu = -g^{\mu\nu} \alpha \nabla_\nu t = -\alpha \nabla^\mu t$$

Using now $g_{\mu\nu}$ and $n_\mu$ we can build a tensor that is orthogonal to $n$, i.e. the metric on $\Sigma$, which takes the form

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$
\[ \gamma_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{ij} \end{pmatrix} \]  

(197)

which means that \( \gamma_{\mu\nu} \) is a purely spatial tensor, i.e. \( \gamma^\mu = 0 \)

\[ \gamma_{\mu\nu}n^\mu = g_{\mu\nu}n^\mu + n_\mu n_\nu n^\mu = n_\nu - n_\nu = 0 \]

(198)

The mixed components of \( \gamma \) are given by

\[ \gamma^\mu_{\ \nu} = g^\mu_{\ \nu} + n^\mu n_\nu = \delta^\mu_{\ \nu} + n^\mu n_\nu \]

(199)

Hence \( \gamma^\mu_{\ \nu}A_\mu = A_\nu \) if \( A_\mu n^\mu = 0 \), i.e. \( \gamma \) can be used to raise/lower indices but only of purely spatial tensors. For fully 4D tensors the metric is needed to raise/lower indices. \( \gamma \) is therefore a spatial projection tensor, i.e. a tensor that given a 4D tensor will return the spatial projection of it on \( \Sigma \).

In a similar way we want an operator that projects "out" of \( \Sigma \), i.e. in the time direction,

\[ N^\mu_{\ \nu} = -n^\mu n_\nu \]

(200)

where of course \( N \cdot \gamma = 0 \). \( N \) is called timelike projection operator

Proof

\[ N^\mu_{\ \nu} \gamma^\nu_{\ \nu} = -n^\mu n_\nu (\delta^\mu_{\ \nu} + n^\mu n_\nu) = -n^\mu n_\mu - n^\mu n_\nu n^\mu n_\nu = 1 - 1 = 0 \quad \text{q.e.d.} \]

In this way we have the mathematical tools not only to spit the manifold \( \mathcal{M} \) in 3 + 1 but also any tensor in it.

Example

Take a 4-vector \( \underline{U} \) and split it in a purely timelike part and in a purely spacelike part

\[ \underline{U} = \underline{U} \cdot (\gamma + N) = \gamma \cdot \underline{U} + N \cdot \underline{U} = A + B \]

where \( A^\mu = (0, A^1, A^2, A^3) \) and \( B^\mu = (B^0, 0, 0, 0) \).
Note: \( \mathbf{n} \) is the unit normal vector to any point in \( \Sigma \). \( \mathbf{n} \) does not represent the direction along which the time coordinate varies. To check this let’s compute
\[
\mathbf{n} \cdot \tilde{\Omega} = \sum n^\mu \Omega_\mu = \frac{1}{\alpha} n^\mu n_\mu = \frac{1}{\alpha} \neq 1
\]

In other words, \( \mathbf{n} \cdot \tilde{\Omega} = \alpha^{-1} \) expresses the fact that the gradient of the time coordinate \( \tilde{\Omega} \) along a given direction \( \mathbf{n} \) is a function!

We need a new timelike 4-vector \( \mathbf{t} \) such that \( \mathbf{t} \cdot \Omega = 1 \) and that acts as the basis vector of the time coordinate. In its most generic definition
\[
\mathbf{t} = \mathbf{e}_t = \alpha \mathbf{n} + \beta
\]
where the second part of the RHS which is purely spatial \( (\beta \cdot \mathbf{n} = 0) \) is called shift vector.

**Proof**
\[
\mathbf{t} \cdot \Omega = (\alpha \mathbf{n} + \beta) \cdot \Omega \Leftrightarrow -\alpha n^\mu + \beta^\mu \frac{\partial \Omega^\mu}{\partial \mathbf{n}^\mu} = 1 + 0 = 1 \quad \text{q.e.d.}
\]

Summarizing the previous results we can say that \( \mathbf{t} \) and \( \Omega \) are always parallel while \( \mathbf{n} \) and \( \mathbf{t} \) are parallel only if \( \alpha = 1 \) and \( \beta = 0 \) (flat spacetime).

It important to notice that \( \mathbf{t} \) is not an unit vector since
\[
\mathbf{t} \cdot \mathbf{t} = g_{tt} = -\alpha^2 + \beta^\mu \beta_\mu = -\alpha^2 + \beta^t \beta_t \neq -1
\]
Using the lapse and shift we can write

\[ g_{tt} = t \cdot \xi = -\alpha^2 + \beta^i \beta_i \]

\[ g_{ti} = t \cdot \gamma = t^\mu \gamma_{\mu i} = t^\mu (g_{\mu i} + n_\mu n_i) = t^\mu g_{\mu i} = (\alpha n^\mu + \beta^\mu) g_{\mu i} \]

\[ = \alpha n_i + \beta^\mu g_{\mu i} = \beta^i g_{ji} = \beta_i \]

so that the line element reads

\[ ds^2 = -(\alpha^2 - \beta^i \beta_i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j \] (203)

with the metric

\[ g_{\mu \nu} = \begin{pmatrix} -\alpha^2 & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix} \quad g^{\mu \nu} = \begin{pmatrix} \alpha^2 & \beta_i / \alpha^2 \\ \beta_i / \alpha^2 & \gamma_{ij} - \beta^i \beta^j / \alpha^2 \end{pmatrix} \] (204)

As a result

\[ n_\mu = (\alpha, 0, 0, 0); \quad n^\mu = \frac{1}{\alpha^2} (1, -\beta^i) \] (205)

Let’s now investigate the physical meaning of the lapse function.

\[ d\tau^2 = -ds^2 = \alpha^2 dt^2 \leftrightarrow d\tau = \pm \alpha dt \] (206)

which means that the lapse function expresses the rate of change of the proper time relative to the change of coordinate time.

We have now three different vector on the hypersurface

\[ v^i = \frac{\partial x^i}{\partial \tau} \]

We need to characterize the spatial part of the (fluid) four-velocity \( U \) measured by Eulerian (normal) observers. We now that the spatial part of \( U \) (which we will call \( v^i \)) is equal to the projection of \( U \) on \( \Sigma \) divided by the projection of \( U \) along \( n \) (space/time). We can therefore write it as

\[ v^i = \frac{\gamma^j \mu u^\mu}{-u_\mu n^\mu} \] (207)

The factor \( -n^\mu u_\mu = \alpha u^i \) can be seen as the Lorentz factor \( W \) since

\[ \alpha u^i = (1 - v^i v_j)^{1/2} = W \] (208)
as in special relativity (exercise).

Proof

A proof of this last equation is easy if we split the 4-velocity $U$ in its covariant and contravariant part

$$v^i = \frac{1}{\alpha} \left( \frac{U^i}{U^t} + \beta^i \right); \quad v_i = \gamma_{ij} v^j = \frac{\gamma_{ij}}{\alpha} \left( \frac{U^i}{U^t} + \beta^i \right)$$

It follows

$$v^i v_i = \frac{1}{\alpha^2} \left( \left( \frac{U^i}{U^t} + \beta^i \right) \gamma_{ij} \left( \frac{U^j}{U^t} + \beta^j \right) \right) = \frac{1}{\alpha^2} \left( \gamma_{ij} \left( \frac{U^i U^j}{(U^t)^2} \right) + 2 \frac{\beta^i U^i}{U^t} + \beta^i \beta_i \right)$$

Whit

$$-1 = U^{\mu} U_\mu = g_{tt} (U^t)^2 + 2 g_{t\mu} U^t U^\mu + U^\mu U_\mu$$

we obtain the relation

$$-1 + (\alpha U^t)^2 = \beta_i \beta_i (U^t)^2 + 2 \beta_i U^t U^i + U^i U_i$$

which in (209) gives

$$v^i v_i = \frac{1}{\alpha (\alpha U^t)^2} (-1 + (\alpha U^t)^2) = \frac{-1 + W^2}{W^2}$$

so that

$$W^2 (1 - v^i v_i) = 1 \Leftrightarrow W = (1 - v^i v_i)^{-1/2} \text{ q.e.d.}$$

In component form

$$v^t = 0; \quad v^i = \frac{\gamma^{i\mu} U^\mu}{\alpha U^t} = \frac{1}{\alpha} \left( \frac{U^i}{U^t} + \beta^i \right)$$

(210)

$$v_i = \beta_i v^i; \quad v_i = \frac{\gamma_{i\mu} U^\mu}{\alpha U^t} = \frac{\gamma_{ij}}{\alpha} \left( \frac{U^j}{U^t} + \beta^j \right)$$

(211)

or, using the Lorentz factor

$$v^i = \frac{U^i}{W} + \frac{\beta^i}{\alpha} = \frac{1}{\alpha} \left( \frac{U^i}{U^t} + \beta^i \right); \quad v_i = \frac{U_i}{W} = \gamma_{ij} \left( \frac{U^j}{W} + \frac{\beta^j}{\alpha} \right)$$

(212)

We recall that in special relativity

$$v^i = \frac{dx^i}{dt} = \frac{dx^i}{d\tau} = \frac{dx^i}{d\tau}$$

So that $v^i = v^i_{SR}$ if $\alpha = 1$ and $\beta = 0$.

It is easy to show that

$$U = W (u + \vec{e})$$

(213)

where the first term is the purely time part and the second term the purely spatial part.
6.1 ADM formulation

Next, we will derive the most famous 3+1 formulation of the Einstein equations, the ADM formulation (from Arowitt, Deser, Misner 1962).

Notes:

- The ADM formulation was not derived for numerical solutions but for the Hamiltonian formulation of the Einstein equations.
- The ADM formulation is seldom used in practice (we’ll explain why).
- Much of the formulation we present comes from the formalism introduced by York (1979).

6.1.1 3D formulation of basic elements

An important first step in the derivation of the ADM equations is the definition of the spatial covariant derivative

$$D_\nu := \gamma^\mu_\nu \nabla_\mu = (\delta^\mu_\nu + n^\mu n_\nu) \nabla_\mu$$

(214)

Just like the 4D covariant derivative is compatible with the four-metric, i.e.\n\n\n$$\nabla_\mu g^{\mu\nu} = 0, \quad \text{so is the 3D covariant derivative with the three-metric}$$

$$D_\mu \gamma^{\mu\nu} = 0$$

(215)

In practice, the spatial covariant derivative is just the result of the projection of the 4D one, i.e.

$$D_\alpha T^\alpha_\beta = \gamma^{\mu}_\alpha \gamma^\rho_\beta \gamma^{\gamma}_\nu \nabla_\mu T^{\nu}_\rho$$

(216)

What is important is that the spatial covariant derivative is based on the 3D Christoffel symbol. We recall that the Christoffel symbols (or connections) are first derivatives of the metric

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} + g_{\beta\gamma,\delta})$$

(217)

so that the corresponding 3D (spatial) objects are

$$^{(3)}\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \gamma^{\alpha\delta} (\gamma_{\delta\beta,\gamma} + \gamma_{\delta\gamma,\beta} + \gamma_{\beta\gamma,\delta})$$

(218)

As for the 4D Christoffel symbols, also the 3D ones follow the same properties: symmetric on lower indices and not proper tensors (do not transform as tensors).

If we want to express a 3+1 decomposition of the Einstein equations, we have to follow the same route as in 4D spacetimes:

$$g_{\mu\nu} \rightarrow \Gamma^\alpha_{\mu\nu} \rightarrow R^\alpha_{\beta\mu\nu} \rightarrow R_{\mu\nu} \rightarrow G_{\mu\nu}$$

Recalling that the 4D curvature tensor is

$$R^\mu_{\nu\beta\alpha} = \partial_\beta \Gamma^\mu_{\alpha\nu} - \partial_\alpha \Gamma^\mu_{\beta\nu} + \Gamma^\mu_{\beta\delta} \Gamma^\delta_{\alpha\nu} - \Gamma^\mu_{\alpha\delta} \Gamma^\delta_{\beta\nu}$$

$$= f(\partial^2 g, (\partial g)^2); \quad [R^\mu_{\nu\beta\alpha}] = L^{-2}$$

(219)
the corresponding spatial 3D tensor can be written as

\[(3) R_{\mu\nu\beta\alpha} = \partial_\beta (3) \Gamma_{\mu\alpha\nu} - \partial_\alpha (3) \Gamma_{\mu\beta\nu} + (3) \Gamma^\delta_{\beta\delta} (3) \Gamma_{\alpha\nu} - (3) \Gamma^\delta_{\alpha\delta} (3) \Gamma_{\beta\nu} \]

which is purely spatial, i.e. \((3) R_{\mu\nu\beta\alpha} n_\mu = 0\).

It is now straightforward to define the spatial 3D Ricci tensor

\[(3) R^\mu_{\alpha\mu\beta} = (3) R_{\alpha\beta} \]

and the spatial 3D Ricci scalar

\[(3) R^\alpha_{\alpha} = (3) \bar{R} \]

It is important not to confuse the 3D Riemann tensor with the corresponding 4D one. While \((3) R^\mu_{\nu\beta\alpha}\) is purely spatial (spatial derivatives of spatial metric \(\gamma\)), \(R^\mu_{\nu\beta\alpha}\) is a full 4D object containing also time derivatives of the full 4D metric \(g\). The information present in \(R^\mu_{\nu\beta\alpha}\) and "missing" in \((3) R^\mu_{\nu\beta\alpha}\) can be found in an other spatial tensor: the extrinsic curvature. As we shall see, this information is indeed describing the time evolution of the spatial metric.

### 6.1.2 Extrinsic curvature

As just said, in going from a 4D curvature tensor \(R\) to a 3D curvature tensor \((3) \bar{R}\) we clearly loose some information, namely the information about how the spatial hypersurface \(\Sigma_t\) is "bent" relative to the embedding 4D spacetime, i.e. the extrinsic curvature. We will understand this concept better with some examples but let’s first learn how to measure the extrinsic curvature \((3) K = \bar{K}\).

It is quite intuitive that a way of measuring the extrinsic curvature is one in which we compare the difference in normals on the hypersurface \(\Sigma_t\). Let \(\nu_P\) be the normal four-vector in \(P\) and \(\nu_Q\) the equivalent in \(Q\). Let \(\nu^P_P\) be the four-vector \(\nu_P\) parallel-transported at \(Q\). We can compare \(\nu^P_P\) and \(\nu_Q\).

Of course we are interested in the projection of \(\delta\nu\) on \(\Sigma_t\)

\[ K = -\gamma \cdot \delta\nu \Leftrightarrow K^\mu_{\nu\alpha} = -\gamma^\alpha_{\mu} \nabla_\nu n_\alpha \]

Let’s open a small digression about the sign of this last equation. The two signs distinguish "convex" and "concave" hypersurfaces:
In our case we have therefore chosen the negative sign. Note that this is not the only way of defining the extrinsic curvature. Other ways to measure it are offered by the "evolution" of the unit normals, i.e. by the acceleration of normal observers. Defining the acceleration of fluid as 
\[ \ddot{a}_\mu = u^\nu \nabla_\nu u_\mu, \]
and the acceleration of normal observers as 
\[ a^\mu = n^\nu \nabla_\nu n_\mu, \]
we can write the extrinsic curvature tensor as
\[ K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu. \tag{224} \]

This definition has a marked physical meaning: Euclidean observers are passive tracers and evolve following the curvature of spacetime, they will not converge if curvature is positive and vice versa. The extrinsic curvature tells us about how the hypersurface is curved ("bent") with respect to the 4D manifold. This is not easy for us to picture, but quite familiar if we restrict to 2D surfaces embedded in an Euclidean 3D space.

**Example** The first example is a plane in Euclidean (flat) \( \mathbb{R}^3 \). Consider a Cartesian coordinate system \( x^i = (x, y, z) \) so that the surface of the plane is given by \( z = 0 \). The scalar function \( t \) defining \( \Sigma_t \) is simply \( z = t \). The spatial metric \( \gamma_{ij} \) induced by \( g_{\alpha\beta} \) is diagonal with components
\[ \gamma_{ij} = \text{diag}(1, 1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
with
\[ ds^2 = dx^2 + dy^2. \]

Obviously the Riemann tensor is zero. The normal three-vector is \( n^i = (0, 0, 1) \) and \( n_i = (0, 0, 1) \), so that \( n \cdot n = 1 \). The extrinsic curvature is then
\[ K_{ij} = -\gamma^k_j \nabla_i n_k = -\gamma^3_j \nabla_i n_3 = 0 \tag{225} \]

since \( \partial_x z = \partial_y z = 0 \). It follows
\[ ^{(3)}R^\mu_{\nu\beta\alpha} = 0 = K \tag{226} \]
The extrinsic curvature of a plane is zero: normal can be parallel transported and do not change.

**Example** We can move on with complexity: let’s consider a cylinder in \( \mathbb{R}^3 \)
It is not difficult to show that \( R^\mu_{\nu\beta\alpha} = 0 \) (you can "cut" a cylinder and lay on a plane without wrinkles). But what about the extrinsic curvature? It is possible to show (exercise) that the extrinsic curvature vanish in the \( e_\nu \) direction but is non-zero in the \( e_\phi \) direction.

**Example**  What about a 2-sphere in \( \mathbb{R}^3 \)?

You will see (exercise) that \( R^\mu_{\nu\beta\alpha} \neq 0 \) and \( K_{ij} \neq 0 \) in \( e_\phi \) and \( e_\theta \) directions.

Let's brush up a bit the concept of **Lie derivative** to discuss yet another way of computing the extrinsic curvature. We had already seen that the covariant derivative of a vector field relative to another one can be computed as

\[
(\mathcal{L}_V U)^\mu = \nabla_V U^\mu - \nabla_U V^\mu = [V, U] \quad (227)
\]

which is in components

\[
(\mathcal{L}_V U)^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu; \quad (\mathcal{L}_V U)_\mu = V^\nu \partial_\nu U_\mu + U^\nu \partial_\nu V_\mu
\]
We can now apply the Lie derivative of the spatial metric relative to the normal vector, i.e. (exercise)

\[ \mathcal{L}_n \gamma_{\mu\nu} = n^\alpha \nabla_\alpha \gamma_{\mu\nu} + \gamma_{\mu\alpha} \nabla_\nu n^\alpha + \gamma_{\nu\alpha} \nabla_\mu n^\alpha = -2K_{\mu\nu} \quad (228) \]

from which we obtain

\[ K_{ij} = -\frac{1}{2} \mathcal{L}_n \gamma_{ij} \quad (229) \]

Recalling now that \( t = \alpha n + \beta \) it follows

\[ \mathcal{L}_n = \frac{1}{\alpha} \mathcal{L}_\alpha n = \frac{1}{\alpha} (\mathcal{L}_t - \mathcal{L}_\beta) \]

\[ \Rightarrow \partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \quad (230) \]

so that

\[ \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j \]

This equation is one of the the so-called evolution equations. Note that this can be seen as a definition of \( K_{ij} \) and a kinematical description of the coordinates, where the extrinsic curvature is equal to the time derivative of the coordinates measured by an Eulerian observer.

To conclude the discussion about the extrinsic curvature, let’s note some properties. The extrinsic curvature is by construction a spatial (obtained with contraction of the spatial metric) and a symmetric (obtained with symmetric derivatives of the unit normal) tensor.

### 6.1.3 Decompose the Einstein equations

In what follows we will use a number of identities derived well before Einstein’s theory of general relativity and that are generic equations of differential geometries resulting from the different possible combinations of projections that can be applied to the Riemann tensor.

- We start with the Gauss-Codezzi equations

\[ \mathcal{G} \cdot \mathcal{G} \cdot \mathcal{G} \cdot \mathcal{G} \cdot \text{Riemann} \]

\[ \Leftrightarrow \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\rho_\delta \gamma^\sigma_\lambda R_{\mu\nu\rho\sigma} = (3) R_{\alpha\beta\delta\lambda} + K_{\alpha\delta} K_{\beta\lambda} - K_{\alpha\lambda} K_{\beta\delta} \quad (232) \]

- Next, we consider the Codezzi-Mainardi equations

\[ \mathcal{G} \cdot \mathcal{G} \cdot \mathcal{G} \cdot \mathcal{D} \cdot \text{Riemann} \]

\[ \Leftrightarrow \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\rho_\delta \gamma^\sigma_\lambda n^\sigma R_{\mu\nu\rho\sigma} = D_\alpha K_{\beta\lambda} - D_\beta K_{\alpha\lambda} \quad (233) \]

85
• Ricci equations

\[ \gamma \cdot \gamma \cdot \gamma \cdot Riemann \]

\[ \Leftrightarrow \gamma^\mu_\alpha \gamma^\nu_\beta n^\delta R_{\alpha\beta\lambda} = (3) R_{\mu\nu} + K K_{\mu\nu} - K^\lambda_{\nu} K_{\mu\lambda} \quad (234) \]

where \( K = K^\mu_\mu \).

Putting things together one obtains the evolution part of the Einstein equations, i.e. the second evolution equation

\[ \partial_t K_{ij} = -D_i D_j \alpha + \beta^k \partial_k K_{ij} + K_{ij} \partial_k \beta^k + k_{il} \partial_j \beta^l + \]

\[ + \alpha (3) R_{ij} + K K_{ij} - 2 K_{il} K^l_j + 4 \pi \alpha [\gamma_{ij} (S - E) - 2 S_{ij}] \quad (235) \]

where the last term correspond to the contribution related to the energy-momentum tensor and it is zero if \( \mathcal{T} = 0 \). The matter quantities are given respectively by

\[ S_{\mu\nu} = \gamma^\alpha_\mu \gamma^\beta_\nu T_{\alpha\beta} \quad (236) \]

which is the spatial part of the energy-momentum tensor,

\[ S_\mu = \gamma^\alpha_\mu n^\beta T_{\alpha\beta} \quad (237) \]

which is the momentum density,

\[ S = S^\mu_\mu \quad (238) \]

which is the trace of \( S \) and

\[ E = n^\alpha n^\beta T_{\alpha\beta} \quad (239) \]

which is the energy density measured by Eulerian observers.

6.1.4 Constraint equations

In order to answer the question on how many evolution equations are there, let’s go back to the Einstein equations. The Einstein tensor is a 4 × 4 matrix with 16 components. Since it is symmetric they reduce to 10 and since they are 2nd-order partial derivative equations (PDEs), the number of 1st-order PDEs rises to 20. We have seen the evolution equations \( \partial_t \gamma_{ij} \) (6 equations) and \( \partial_t K_{ij} \) (6 equations) which are 12 in total. There are still 8 equations that we have not yet accounted for. There are some contractions that we have not yet considered. With the condition

\[ \gamma \cdot \gamma \cdot \gamma \cdot Riemann \Leftrightarrow \gamma^{\alpha\mu} \gamma^{\beta\nu} R_{\alpha\beta\mu\nu} = 2 G_{\mu\nu} n^\mu n^\nu \]

we obtain the first equation, the Hamiltonian constraint,

\[ (3) R + K^2 - K_{ij} K^{ij} = 16 \pi E \quad (240) \]
while other three equations, the momentum constraints, are given by
\[ \gamma \cdot \nabla \cdot \mathbf{G} \Leftrightarrow \gamma^{\alpha \mu} n_{\mu} R_{\nu} ^{\nu} \nu = D^\alpha K - D_\mu K^{\alpha \mu} \]
so that
\[ D_j (K^{ij} - \gamma^{ij} K) = 8\pi S^i \tag{241} \]

It is important to underline that equations (240) and (241) do not have time derivatives: they are constraint equations. The 12 evolution equations and the 4 constraint equations are also known as the ADM equations.

### 6.2 ADM vs Maxwell

The appearance of these two classes of equations can be found in Electrodynamics: Maxwell equations in fact have a very similar split. The evolution equations are
\[ \partial_t E = \nabla \times B - 4\pi j \Leftrightarrow \partial_t E_i = \epsilon_{ijk} \partial_j B_k - 4\pi j_i \]
\[ \partial_t B = -\nabla \times E \Leftrightarrow \partial_t B_i = -\epsilon_{ijk} \partial_j E_k \]
while the constraint equations are
\[ \nabla \cdot B = 0 \Leftrightarrow \partial_i B^i = 0 \]
\[ \nabla \cdot E = 4\pi \rho_e \Leftrightarrow \partial_i E^i = 4\pi \rho_e \]

The analogies between the ADM and the Maxwell equations will help us understand some of the problems associated with the ADM equations and suggest possible solutions. In practice the ADM equations have not been used in 3D applications and their use has been abandoned when it has become clear that they are weakly hyperbolic. To appreciate the implications of this statement we need a small digression.

#### 6.2.1 Hyperbolicity

A large class of equations in mathematical physics (e.g. EFEs, equations of HD and MHD) can be written in a compact form as
\[ \partial_t \mathbf{U} + A \nabla \mathbf{U} = \mathbf{S} \Leftrightarrow \partial_t U_j + (A^i)_{jk} \nabla_j U_k = S_j \tag{242} \]
where \( \mathbf{U} = \{U_1, U_2, \ldots, U_j\} \) is the state vector and \( \mathbf{S} = \{S_1, S_2, \ldots, S_j\} \) is the source term. \( A \) is a matrix of coefficients the index \( i \) specifies the different matrices, one for each direction. The properties of the system (242) depend on the properties of \( A \) and \( S \):

- Is \( a_{jk} = \text{const} \) an element of \( A \) and \( S_j = \text{const} \), then (242) is a linear system of equations with constant coefficients.
- Is \( a_{jk} = a_{jk}(x,t) \) and \( S_j = S_{jk}(x,t) \), then (242) is a linear system with variable coefficients.
- Is \( A = A(U) \), then (242) is a non linear system (often referred to as quasi-linear).
More importantly, the system (242) is said to be (strongly) hyperbolic if $A$ is diagonalizable with a set of real eigenvalues $\lambda_1, \ldots, \lambda_N$ and a set of $N$ linearly independent right eigenvectors, i.e. if $\Lambda := R^{-1}AR = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ where $R$ is a matrix of right eigenvectors $R^{(i)}$, so that $AR^{(i)} = \lambda_i R^{(i)}$ with $\lambda_i \in \mathbb{R}$ as real eigenvalues.

- (242) is said to the strictly hyperbolic if $\lambda_i$ are real and distinct.
- (242) is said to be symmetric hyperbolic if $A$ is symmetric, i.e. $A = A^T$.
- (242) is said to be weakly hyperbolic if $A$ is not diagonalizable

Example of hyperbolic equations are

- Advection equation $\partial_t u + v \partial_x u = 0$
- Wave equation $\partial^2_t u + v^2 \partial^2_x u = 0$
- Hydrodynamic equations (inviscid)
- Einstein equations

The important of hyperbolic equations is strictly related with that of well posedness of the Cauchy initial-value problem. Let $U(x, 0)$ be the initial data, $U(x, t)$ the solution of the set (242) at time $t$, then (242) is well posed is

$$\|U(x, t)\| \ll ke^{at}\|U(x, 0)\|$$

with $k, a \in \mathbb{R}$ as constants. In other words the solution is always bounded by some exponential of the initial data.

An important theorem of hyperbolic systems states: is (242) a hyperbolic set of equations, then (242) is well posed. The opposite implication is not true. It follows that a weakly hyperbolic system is not guaranteed to be well-posed and indeed the numerical solution leads to the growth of unstable modes ("codes crash"). In the case of the ADM equations the weak hyperbolicity comes from the mixed derivatives in the Ricci tensor for the evolution of the extrinsic curvature (while one wishes to have only diagonal second derivatives), i.e. see the fourth term in equation (235) and consider $(^3\!\!R_{ij}$ with $\gamma^l \gamma^m \partial_i \partial_j$.

### 6.3 BSSNOK formulation

There are a number of ways around this problem and the easiest way to understand how this works is to look at the Maxwell equations. Let’s introduce the vector potential $A := \sum \times B$ with components $A_\mu = (-\phi, A_i)$. The the Maxwell evolution equations can be written as

$$\partial_t A_i = -E_i - D_i \phi$$
$$\partial_t E_i = -D_i^j D_j A_i + D_i D^j A_j - 4\pi j_i$$

(244)
(245)

to be compared with the ADM evolution equations. Note: of course in classical EM $D_i \leftrightarrow \partial_i$ but we are keeping the $D$-notation to highlight the analogy with GR.

The term $D_i D^j A_j$ breaks the hyperbolicity and we want to get rid of this term. There are different ways to do it.
1. Lorentz gauge:

\[ \partial_t \phi = -D^j A_j \]  

(246)

This does the job because by taking another time derivative of \( \partial_t A_i \) and making use of equation (244) equation (245) becomes

\[ D^j D_j A_i - D_i D^j A_j - \partial_t^2 A_i = D_i (\partial_t \phi) - 4\pi j_i \]

\[ D^j D_j A_i - \partial_t^2 A_i = -4\pi j_i \leftrightarrow \Box A_i = 4\pi j_i \]

(247)

where \( \Box \) is the d’Alembertian. Equation (247) is an hyperbolic equation and hence well-posed.

2. The second route doesn’t involve a gauge but an auxiliary quality. Let’s define a scalar function \( \Gamma := D_i A_j \) such that equation (244) can be written as

\[ \Box A_i = -D_i \Gamma - D_i \partial_t \phi + 4\pi j_i \]

(248)

Written in this way equation (248) is again hyperbolic because the principal part (i.e. \( \Box A_i \)) is the same and the “disturbing” term appears now as a source of the RHS \( (D_i \Gamma) \).

To fix the ADM equations and remove weak hyperbolicity we do something very similar, i.e. we introduce new quantities. Before doing this, we need also some other ingredients.

1. Introduce a conformally related metric \( \tilde{g}_{\mu\nu} \), i.e.

\[ g_{\mu\nu} \leftrightarrow \tilde{g}_{\mu\nu} = \phi^n g_{\mu\nu} \]

where \( \phi \) is the conformal factor so that

\[ ds^2 = \phi^n g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{\mu\nu} dx^\mu dx^\nu \]

A conformal metric allows us to set some additional conditions on the properties of the determinant of the corresponding 3-metric. In particular \( \tilde{\gamma}_{ij} = \phi^2 \gamma_{ij} \) or \( \tilde{\gamma}^{ij} = \phi^{-2} \gamma^{ij} \) so that we can impose the volume element to be \( \tilde{\gamma} := \text{det}(\tilde{\gamma}_{ij}) = 1 \) and the conformal factor is given by \( \phi = (\text{det}(\gamma_{ij}))^{-1/6} = \gamma^{-1/6} \).

2. Introduce the conformally related Christoffel symbols (or Christoffel symbols of conformal three-metric)

\[ \tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + 2(\delta^i_j \partial_k \ln \phi + \delta^i_k \partial_j \ln \phi - \gamma_{jk} \gamma^{il} \partial_l \ln \phi) \]

3. Introduce the conformally related trace-free extrinsic curvature

\[ \tilde{A}_{ij} = \phi^2 A_{ij} = \phi^2 \left( K_{ij} - \frac{1}{3} \gamma_{ij} K \right) \]

(249)

with \( \tilde{A}^{ij} = \phi^{-2} A^{ij} \), so that

\[ \tilde{A}_{ij} \gamma^{ij} = \tilde{A}^i_i = \phi^2 \left( K^i_i - \frac{1}{3} \gamma_{ij} \gamma^{ij} K \right) = 0 \]

(250)

In other words the conformal extrinsic curvature is traceless.
4. Introduce additional variables to separate mixed derivatives: the "Gammas"

$$\tilde{\Gamma}^i := \tilde{\gamma}^{jk} \partial_j \tilde{\gamma}^i = \tilde{\gamma}^{ij} \partial_j \tilde{\gamma}^i$$  (251)

The resulting set of equations is then

$$\partial_t \tilde{\gamma}^{ij} = -2\alpha \tilde{A}^{ij} + 2\tilde{\gamma}^{k} (\partial_j \tilde{\gamma}^i) - 3 \tilde{\gamma}^{ij} \partial_k \beta^k + \beta^j \partial_k \tilde{\gamma}^i$$  (252)

$$\partial_t \tilde{A}^{ij} = \phi^2 [\tilde{R}^{ij} + \alpha (3) R^{ij} - 8\pi S^{ij}]^{TF} + \beta^i \partial_k \tilde{\gamma}^j + \ldots$$  (253)

$$\partial_t \phi = \frac{1}{3} \phi^2 - \frac{1}{3} \partial_i \beta^i + \beta^k \partial^k \phi$$  (254)

$$\partial_t K = -D_i D_j \alpha + \alpha [\tilde{A}^{ij} \tilde{\gamma}^i + \ldots$$  (255)

$$\partial_t \tilde{\Gamma}^i = \tilde{\gamma}^{ik} \partial_j \partial_k \beta^i + \frac{1}{3} \tilde{\gamma}^{ik} \partial_k \beta^j + \ldots$$  (256)

where the "TF" indicates that the trace-free part of the brackets is used, i.e.

$$(3) R^{ij}^{TF} \rightarrow (3) R^{ij} - \frac{1}{3} \gamma^{ij} (3) R$$

This equations are those normally used and are referred to as the BSSNOK formulation or conformally traceless. When comparing with the ADM equations we have clearly gained 5 more equations ($\partial_t \phi$, $\partial_t K$, $\partial_t \tilde{\Gamma}^i$) but the system is now hyperbolic and indeed well behaved in numerical simulations. The additional computational costs are 12+5-2=15 variables (against the 12 variables in the ADM equations) since $\tilde{\gamma}^{ij}$ and $\tilde{A}^{ij}$ are traceless or with known trace.

Similarly the Hamiltonian constraint equation is then

$$\begin{align*}
(3) R &= \tilde{A}^{ij} \tilde{A}^{ij} - \frac{2}{3} K^2 + 16\pi E 
\end{align*}$$  (257)

Proof

$$(3) R + K^2 - K_{ij} K^{ij} = 16\pi E$$

$$= (3) R + K^2 - \left( \tilde{A}^{ij} + \frac{1}{3} \gamma_{ij} K \right) \left( \tilde{A}^{ij} + \frac{1}{3} \gamma_{ij} K \right)$$

$$= (3) R + K^2 - \left( \tilde{A}^{ij} \tilde{A}^{ij} + \frac{2}{3} \tilde{A}^{i} K + \frac{1}{9} 3K^2 \right)_{=0}$$

$$(3) R + K^2 - \tilde{A}^{ij} \tilde{A}^{ij} - \frac{1}{3} K^2$$

$$(3) R - \tilde{A}^{ij} \tilde{A}^{ij} + \frac{2}{3} K^2$$ q.e.d.

and the momentum constraint equations are

$$D_j (\tilde{A}^{ij} - \frac{2}{3} \phi^{2} \gamma^{ij} K) = 8\pi S^i$$  (258)
Proof

\[ D_j(K^{ij} - \gamma^{ij} K) = 8\pi S^i \]
\[ = D_j(\tilde{A}^{ij} + \frac{1}{3}\gamma^{ij} K - \gamma^{ij} K) \]
\[ = D_j(\tilde{A}^{ij} - \frac{2}{3}\gamma^{ij} K) \] q.e.d.

How are the constraints handled? These are 3D non-linear elliptic equations to be solved on each slice and very expensive to solve. Solving one of these is more expensive than computing the full set of evolution equations. In practice the constraint equations are monitored as a increase of the quality of the solution.

6.4 CCZ4 formulation

A property that the BSSNOK does not have is that of damping the constraint violations, i.e. the property of reducing the violations if a violation is for some reason introduced.

Let’s go back to EM

\[ \nabla \cdot B = 0 \Leftrightarrow \partial_t B^i = 0 \]

Let \( \Psi = \partial_i B^i \) and assume it is not zero initially.

\[ \partial_t \Psi = \partial_i \partial_i B^i = \partial_i \partial_t B^i = -\partial_t \epsilon^{ijk} \partial_j E_k = -\epsilon^{ijk} \partial_i \partial_j E_k = 0 \Leftrightarrow \Psi = \text{const} \]

So if \( \Psi = 0 \) initially, it would be equal to zero at all times but if \( \Psi \neq 0 \) initially, it would be \( \neq 0 \) at all times. This is not a desirable feature which can be conquered if we write

\[
\begin{align*}
\partial_t B^i &= -\epsilon^{ijk} \partial_j E_k + \gamma^{ij} \partial_j \Psi \\
\partial_t \Psi &= -a_1 \partial_i B^i - a_2 \Psi
\end{align*}
\]

Clearly we return to the initial system if

\[ \Psi(x, 0) = 0 = \partial_t \Psi(x, 0) \]

However, if \( \partial_i B^i \neq 0 \) then the action of the scalar field will be that of driving the solution exponentially fast toward \( \Psi = 0 = \partial_t B^i \).
This spirit is behind new formulations of the EFEs that have been derived over the last couple of years. These are the Conformal Covariant Z4 (CCZ4) and the ZAc formulation. Details can be found in the book.

6.5 Gauge conditions

Let’s go back to the counting of the equations: there are 20 equations of 1st-order in time given by the (10 equations of 2nd-order in time of the) Einstein equations and we defined 16 equations by the ADM formulation. The 4 missing equations represent the gauge freedom inherent to GR and corresponds to the freedom in specifying the lapse function $\alpha$, slicing condition, and the shift vector $\beta^i$, spatial gauge condition. The arbitrariness of the gauge implies that the large majority of the quantities computed in a numerical relativity simulation are "gauge dependent", i.e. they will depend on the gauge used. This however does not prevent the existence and calculation of "gauge independent" quantities, e.g. gravitational radiation, masses of spacetimes, etc. Arbitrariness of gauge choice is an advantage and handicap at the same time. Good gauge conditions make difference between stable evolution and crash.

6.5.1 Requirements of good gauge choices

1. If singularities are present or develop on a slice, these are avoided: singularity avoiding property.  
   Example: Oppenheimer-Sneider collapse (analytic solution of the collapse of a collisionless cloud of particle)  
   In this solution a physical singularity is produced when all fluid shells reach the center. This happens at the same time for all shells. An event horizon starts to grow from zero size and reaches the asymptotic value when the cloud surface crosses the $r = 2M$ surface.
There are several different ways of slicing this spacetime. The simplest is called \textbf{geodesic slicing}, this is the slicing of freely falling particles: $\alpha = 1$ and $\beta^i = 0$.

This approach has at least two undesired features.

- Code crashes when the singularity develops (coordinates coincide with position of spacetime with singular curvature). This happens before any radiation reaches distant observer.
- Nicely spaced coordinates end up concentrated and resolution is bad in outer regions.

How can we improve on this and avoid the pathologies mentioned above? This slicing should be such that the coordinate time is slowed down near the singularity. The coordinate should be "pushed out" such that they do not fall in.

- In this way the time is "slowed down" near the singularity. It is as if you never crash against a wall towards which you are accelerating just because you reach it even more slowly.
- All the radiation that needs to reach a distant observer has the time to do so.

How do we impose this singularity avoiding condition? It’s not difficult to show that the condition

$$ K = 0 = \gamma^{ij} K_{ij} $$ (259)
leads to coordinate volume elements that are **maximal**, i.e. they are found at the maximum with small variation in time. The volume element is defined as

\[ V = \int \sqrt{\det(\gamma_{ij})} d^3x \]

and

\[ \partial_t V \propto \int \alpha K \sqrt{\gamma} d^3x \]

so that the condition \( K = 0 \) implies \( \partial_t V = 0 \).

So what we want is not only that \( K = 0 \) but also that \( \partial_t K = 0 \)

\[ K = 0 = \partial_t K \] (260)

This condition is called maximal slicing and requires that Eulerian observers in free fall do not “focus” where extrinsic curvature increases.

A bit of algebra then leads to show that the maximal slicing condition is equivalent to

\[ \gamma^{ij} \partial_i \partial_j \alpha = D^2 \alpha = \alpha [K_{ij} K^{ij} + 4\pi(E + S)] \] (261)

This is an elliptic equation that needs to be solved on every spatial slice \( \Sigma_t \). As for previous elliptic equations we have discussed before, this equation is computationally too expensive to be solved on each \( \Sigma_t \). In practice, hyperbolic (i.e. time evolving) slicing conditions are used that a condition similar to the maximal slicing condition. A popular choice is the so-called **Bona-Masso family** of slicing conditions

\[ \partial_t \alpha - \beta^k \partial_k \alpha = -f(\alpha)(K - K_0)\alpha^2 \] (262)

where \( f(\alpha) > 0 \) and \( K_0 := K(t = 0) \) is the initial trace.

Different values of \( f \) allows one to obtain several important and well known slicing conditions. **Examples**

- \( f = 1 \) implies \( \partial_t \alpha \propto -K \alpha^2 \) and is called **harmonic slicing**
• \( f = q/\alpha \), \( q \in \mathbb{N} \)
This condition then leads to a lapse function for \( q = 2 \) and \( \beta = 0 \)
\[
\alpha = 1 + \log \gamma
\]
and in called 1 + log slicing.

• If \( f \to \infty \) the Bona-Massè family tends to the maximal slicing condition.

Similar considerations apply also for the spatial gauge conditions, i.e. for the conditions to be imposed on the shift vector.

2. The second requirement of an optimal gauge condition is that if coordinate conditions take place (e.g. collapse of coordinates stretching or rotation) these are counteracted to minimize the distortion. Mathematically this is not difficult to impose. Consider the metric strain tensor which measures size and slope of volume element

\[
\Theta_{ij} = \frac{1}{2} \mathcal{L}_i \gamma_{ij} = \frac{1}{2} (\alpha \mathcal{L}_i + \mathcal{L}_j) \gamma_{ij} = -\alpha K_{ij} + \frac{1}{2} \mathcal{L}_i \gamma_{ij}
\]

Compute the contraction \( \Theta_{ij} \Theta^{ij} \) over the spatial slide \( \Sigma_t \) and minimizing its variation leads to

\[
D_j \Theta^{ij} = 0 \Rightarrow D^2 \beta^i + D^i D_j \beta^j + R^i \beta^j = 2 D_j (\alpha K^{ij})
\]

which is called minimal strain condition. This are perfectly reasonable but they are three elliptic equations to be solved on \( \Sigma_t \).

The situation does not improve considerably when considering the distortion tensor, which measures only the change in shape of volume elements and not the change in size. The tensor is defined as the trace-free part of \( \Theta_{ij} \)

\[
\Sigma_{ij} := \Theta_{ij} - \frac{1}{3} \gamma_{ij} \Theta_{kl} \gamma^{kl}
\]

Working in a similar manner, integrating the positive contraction of \( \Sigma_{ij} \Sigma^{ij} \) and minimizing its variation one obtains

\[
D^i \Sigma_{ij} = 0 \Rightarrow D^2 \beta^i + \frac{1}{3} D^i D_j \beta^j + R^i \beta^j = 2 D_j (\alpha A^{ij})
\]

which is called minimal-distortion shift condition.

Once again we have obtained the required condition but at the expense of a new set of three elliptic equations. This is computationally not feasible with present solves and computes and alternative approaches are usually employed.

In practice, most 3 + 1 simulations use the following hyperbolic conditions

\[
\begin{align*}
\partial_i \beta^i - \beta^j \partial_j \beta^i & = \frac{3}{4} B^i \\
\partial_i B^i - \beta^j \partial_j \beta^i & = \partial_i \Gamma^i - \beta^j \partial_j \Gamma^i - \eta B^i
\end{align*}
\]
where $B^i$ is an auxiliary variable. This gauge condition is called "Gamma-driver" shift condition and it essentially changes the shift in such a way as to "freeze" the evolution of the Gammas, i.e. such that $\partial_t \Gamma^i \approx 0$. In turn, this amounts to a conditions that tends to minimize changes in the distortion tensor (exercise). The Gamma-driver works well in most cases but it still represents a tunable prescription. For example, the term $\eta$ is called damping term and is used to damp large oscillations in the shift (think of a BH moving on the grid). $\eta = 2/M$, where the $M$ is the ADM mass of the spacetime, but it can be chosen also to be a function of space and time. In other words: there is a lot of flexibility in the definition of the gauges and a certain trial and error is necessary most difficult cases.

### 6.6 Initial data

Let’s go back to the constrain equations

$$
\begin{align*}
(3) & \quad R + K^2 - K_{ij}K^{ij} = 16\pi E \\
D_j(K^{ij} - \gamma^{ij}K) & = 8\pi S^i
\end{align*}
$$

They can be seen as the solution of the Einstein equations at a given time $t = 0$ and hence provide the "initial data" for $t > 0$. The evolution equations $\partial_t \gamma_{ij}$ and $\partial_t K_{ij}$ will provide the evolution of the initial data for subsequent times.

We had already discussed that the evolution equations are $6 + 6 = 12$ and so it is unclear how the 4 constrain equations can provide the initial data for the system $\{\gamma_{ij}, K_{ij}\}$.

We have seen that 4 of the 8 remaining equations can be specified after a choice of the gauges. The 4 remaining and unspecified equations represent the intrinsic degree of freedom in the theory.

Hence the problem is how to use the constrain equations to determine the values of $\{\gamma_{ij}, K_{ij}\}$ that are not free. In particular we need a procedure that allow us to select the "longitudinal" part of the initial fields, which can be solved on the slice $\Sigma_t$, from the "transversal" parts of the initial fields, that being dynamical cannot be constrained on $\Sigma_t$.

In general the non-linear nature of the Einstein equations prevents from this distinction, but a procedure can be prescribed in which $\gamma_{ij}$ and $K_{ij}$ are decomposed in quantities that are constrained and in others that are freely specifiable. This procedure is called York-Lichnerowicz conformal decomposition and can be seen as the attempt to split the longitudinal and the transverse part of the equations.

Start from conformal decomposition of the metric (note that this different from the decomposition met in the BSSNOK equations, where we had required $\tilde{\gamma} = 1$)

$$
\tilde{\gamma}_{ij} = \psi^{-4}\gamma_{ij}
$$

where $\tilde{\gamma} = det(\tilde{\gamma}_{ij}) = \psi^{-12}\gamma$. Using now the trace-free part of the extrinsic curvature

$$
A^{ij} = K^{ij} - \frac{1}{3}\gamma^{ij}K
$$

we can rewrite the Hamiltonian constraint as

$$
8\tilde{D}\psi - \tilde{R}\psi + \psi^5 \left( A^{ij}A_{ij} - \frac{2}{3}K^2 \right) + 16\pi \psi^5 E = 0
$$

(271)
where $\bar{D}$ is the covariant derivative relative to $\bar{\gamma}_{ij}$ and $\bar{D}^2 := \bar{\gamma}^{ij} \bar{D}_i \bar{D}_j$. Similarly, the momentum constrain equation can be written as

$$D_j \bar{A}^{ij} - \frac{2}{3} \bar{D}^i K - 8\pi S^i = 0 \quad (272)$$

Next, if $U^{ij}$ is a generic symmetric and trace-free tensor we decompose it into a "transverse", i.e. divergence-free, part and a longitudinal part, i.e.

$$U^{ij} = U^i_{TT} + U^i_L = U^i_{TT} + (\bar{L}W)^{ij} \quad (273)$$

where $U^i_{TT}$ is divergence- and trace-free, i.e. $\bar{\gamma}^{ij} U^i_{TT} = 0 = \bar{D}_j U^i_{TT}$ and where $\bar{L}$ is the longitudinal operator (or conformal Killing operator) acting on the generic vector $W^i$, such that

$$(\bar{L}W)^{ij} := 2D^i W^j - \frac{2}{3} \gamma^{ij} D_k W^k \quad (274)$$

Using the conformal metric $\bar{\gamma}_{ij}$ it is possible to write the conformal trace-free extrinsic curvature as

$$\bar{A}^{ij} = \psi^{10} A^{ij} \quad \bar{A}_{ij} = \psi^2 A_{ij} \quad (275)$$

so that the momentum constrain equation becomes

$$D_j \bar{A}^{ij} - \frac{2}{3} \psi^6 \bar{D}^i K - 8\psi^{10} S^i = 0 \quad (276)$$

Next, split $\bar{A}^{ij}$ is its TT and longitudinal part

$$\bar{A}^{ij} = \bar{A}^i_{TT} + (\bar{L}W)^{ij} \quad (277)$$

where $W$ is the conformal Killing operator relative to $\bar{\gamma}$, so that

$$D_j \bar{A}^{ij} = D_j \bar{A}^i_{TT} + D_j (\bar{L}W)^{ij}$$

$$= 0 + \bar{D}^2 W^i + \bar{D}_j \bar{D}^i W^j - \frac{2}{3} \bar{D}^i \bar{D}_j W^j$$

$$= \bar{D}^2 W^i + \frac{1}{3} \bar{D}^i \bar{D}_j W^j + \bar{R}^i_j W^j = (\bar{\Delta} \bar{L} W)^i$$

where $\bar{\Delta}_L$ is the Laplacian vector and $\bar{R}^i_j$ the conformal Ricci tensor.

Introduce now a symmetric tensor $\bar{M}^{ij}$ such that $\bar{M}^{ij}_{TT} = \bar{A}^{ij}_{TT}$, that is, $\bar{M}^{ij} = \bar{M}^{ij}_{TT} + (\bar{L}Y)^{ij}$, so that the $TT$ part is

$$\bar{M}^{ij}_{TT} = \bar{M}^{ij} - (\bar{L}Y)^{ij} = \bar{A}^{ij}_{TT}$$

so that

$$\bar{A}^{ij} = \bar{A}^{ij}_{TT} + (\bar{L}W)^{ij}$$

$$= \bar{M}^{ij} - (\bar{L}Y)^{ij} + (\bar{L}W)^{ij}$$

$$= \bar{M}^{ij} - (\bar{L}V)^{ij}$$

97
where $\bar{V}^i = W^i - Y^i$ and we have used the linearity of the conformal Killing operator

$$\bar{\mathcal{L}}(W - Y)^{ij} = (\bar{\mathcal{L}}W)^{ij} - (\bar{\mathcal{L}}Y)^{ij}$$

Collecting things, the constrain equations (271) and (272) can be written as

$$8 \bar{D}^2 \Psi - R\psi + \Psi^{-7} \bar{A}^{ij} \bar{A}_{ij} - 2 \frac{\Psi^5}{3} K^2 + 16\pi \Psi^6 E = 0 \quad (278)$$

$$\bar{\Delta}_\Sigma V^i + \bar{D}_i \bar{M}^{ij} - 2 \frac{\Psi^6}{3} \bar{D}^i K - 8\pi \Psi^{10} S^i = 0 \quad (279)$$

This constrain equations have 4 constrained variables: $\Psi, V^i$ and a set of freely specifiable variable $\bar{\gamma}_{ij}, K, \bar{M}_{ij}, Y^i, E, S^i$. Once $\Psi$ and $V^i$ are computed, it is possible to reconstruct

$$W^i = V^i - Y^i$$

$$\bar{\gamma}_{ij} = \Psi^4 \bar{\gamma}_{ij}$$

$$K^{ij} = \Psi^{-10} \bar{A}^{ij} + \frac{1}{3} \gamma^{ij} K = \Psi^{-10} [\bar{A}^{ij}_T + (\bar{\mathcal{L}}W)^{ij}] + \frac{1}{3} \gamma^{ij} K$$

and fully specify $\bar{\gamma}_{ij}, K_{ij}$ on $\Sigma_t$.

Notes:

- It is possible to introduce conformally related quantities also for the matter: $E \rightarrow \bar{E} := \Psi^8 E$ and $S^i \rightarrow \bar{S}^i := \Psi^{10} S^i$.

- Equations (278) and (279) are generally coupled but can be decoupled for constant mean curvature slicing, i.e. $\bar{D}_i K = 0$. In this case one first solves the momentum constraints to get $V^i \rightarrow \bar{A}^{ij}$ and so to the Hamiltonian constraint for $\Psi$.

- The Hamiltonian constraint further simplifies if $\bar{D}_i K = 0$ and $\bar{\gamma}_{ij} = \delta_{ij}$: conformally flat metric. This choice suppresses the radiative degrees of freedom (no GWs an the initial slice) but (278) becomes $(\bar{R} = 0)$

$$8 \bar{D}^2_{flat} \Psi \Psi^{-7} \bar{A}^{ij} \bar{A}_{ij} - 2 \frac{\Psi^5}{3} K^2 + 16\pi \Psi^6 E = 0$$

where $\bar{D}^2_{flat}$ is the Laplacian operator in flat spacetime.

- An additional simplification comes from considering time-symmetric slices, $K_{ij} = 0$. In this case the momentum constraints are automatically satisfied and only the Hamiltonian constraint need to be solved.

Conformally flat, time-symmetric initial data is the one normally used in numerical simulations.