General-Relativistic Radiative Transfer

Ziri Younsi

4th November 2014

Outline

- Background to ray-tracing around black holes
- General-Relativistic (GR) Radiative Transfer (RT) formulation
- GRRT for a geometrically thin and optically thick accretion disk
- Applying GRRT to 3D accretion tori: optically thick, optically thin and quasi-opaque (translucent)
- Compton scattering in GR
- Conclusions and future work

Black Hole Geodesics

- The Kerr (spinning) black hole is an exact solution of the Einstein field equations
- From the metric we may construct the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(g_{tt}\dot{t}^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2\right)$$

- From the Euler-Lagrange equations we may obtain the relevant ODEs which may be solved, given appropriate initial conditions, yielding the geodesics of photons and particles

Black Hole Geodesics

- Four constants of motion (μ, *E*, L_z, *Q*) allow problem to be reduced to one of quadratures, yielding 4 ODE's:

$$\dot{t} = E + \frac{2Mr}{\Sigma\Delta}\left[(r^2+a^2)E - aL_z\right] ,$$

$$\dot{r}^2 = \frac{\Delta}{\Sigma}\left(\mu + E\dot{t} - L_z\dot{\phi} - \Sigma\dot{\theta}^2\right) ,$$

$$\dot{\theta}^2 = \frac{1}{\Sigma^2}\left[Q + \left(E^2+\mu\right)a^2\cos^2\theta - L_z^2\cot^2\theta\right] ,$$

$$\dot{\phi} = \frac{2aMrE + (\Sigma - 2Mr)\,L_z \mathrm{cosec}^2\theta}{\Sigma\Delta}$$

$$\Sigma \equiv r^2 + a^2\cos^2\theta$$

$$\Delta \equiv r^2 - 2Mr + a^2$$

$$\mu \equiv p_\alpha p^\alpha$$

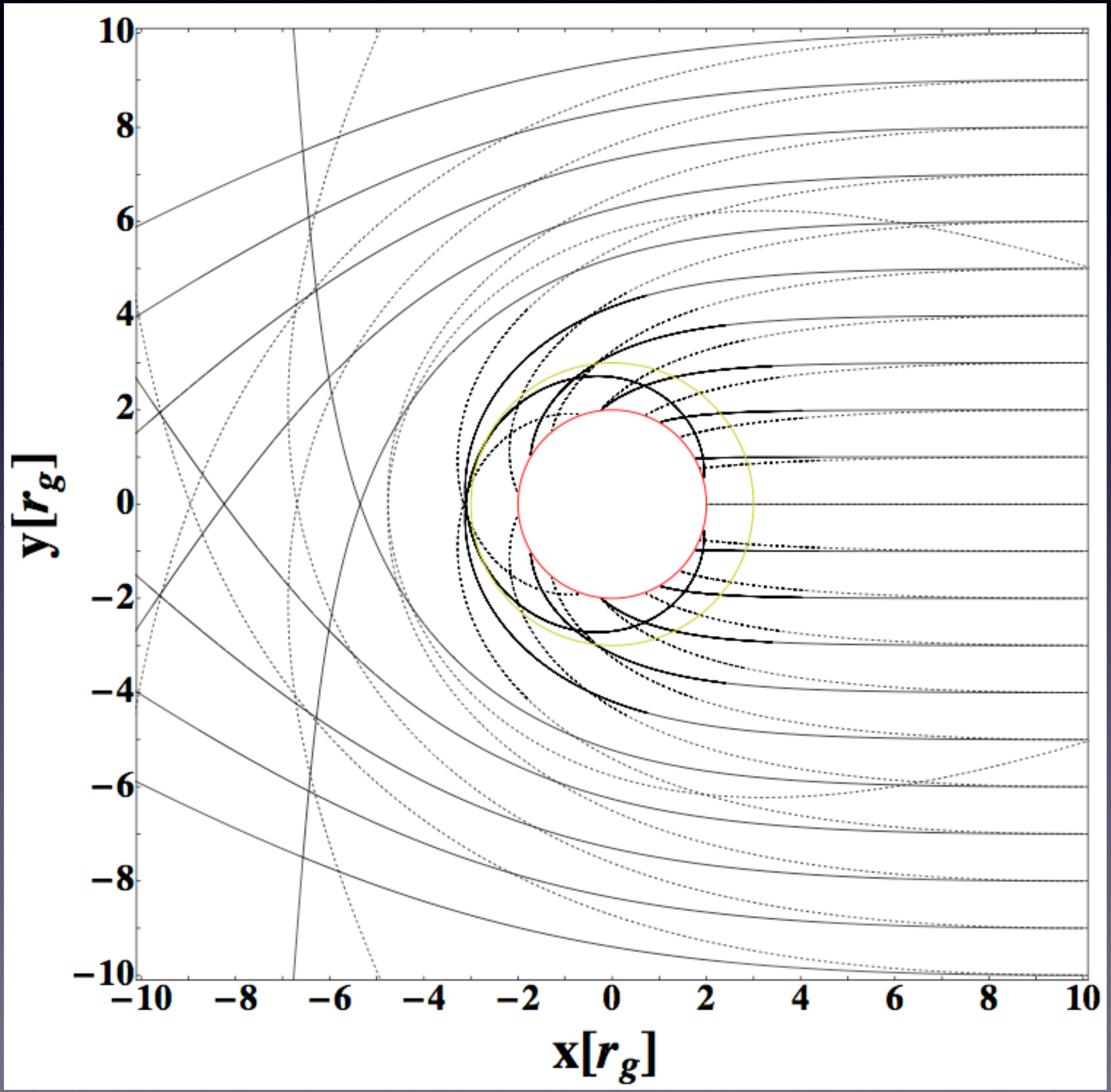
- However, square roots in the red ODEs for *r* and *θ* introduce ambiguity in their signs at turning points

Black Hole Geodesics

- At the expense of solving 2 additional ODEs we may circumvent this problem:

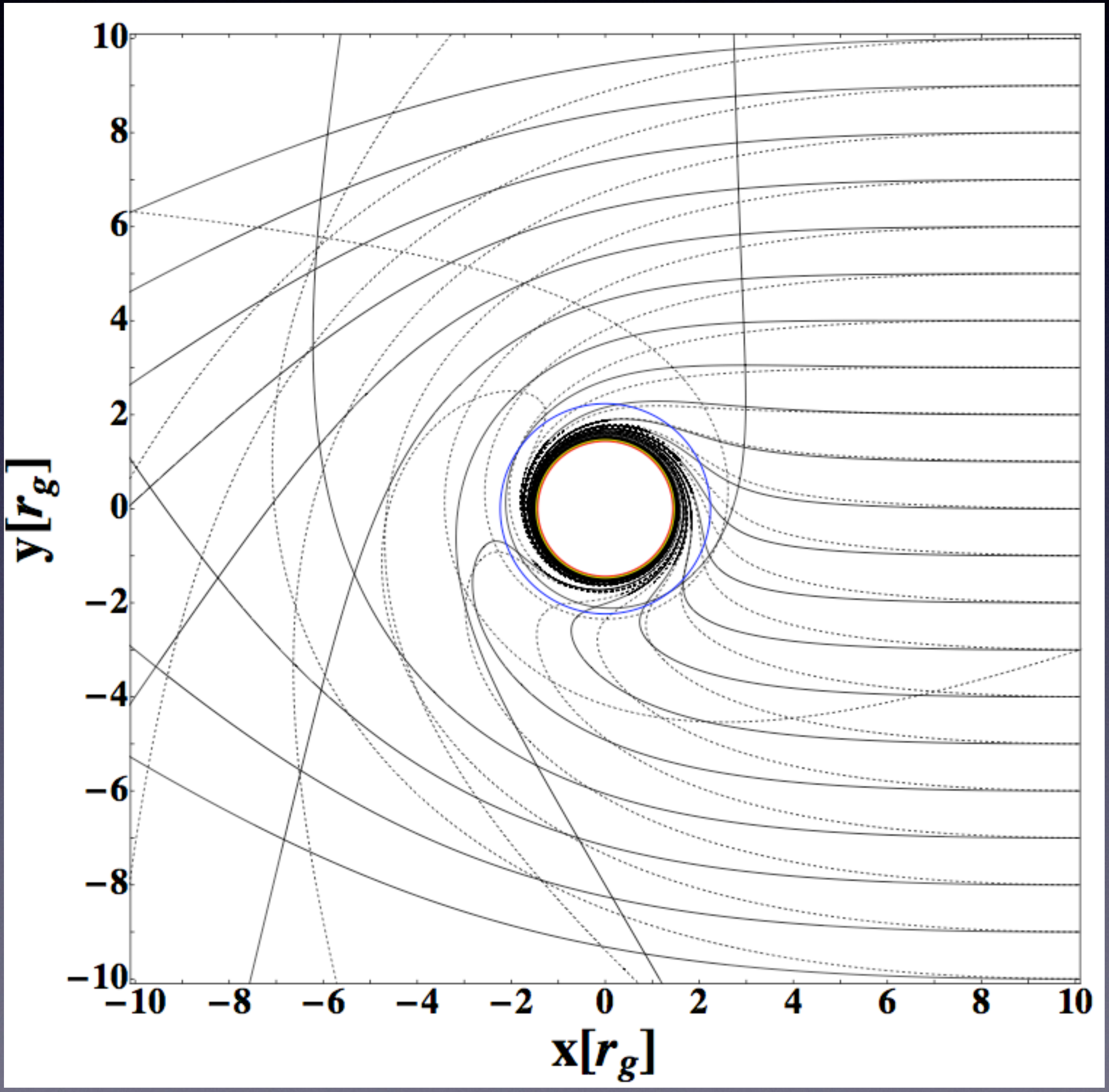
$$
\begin{aligned}
\ddot{r} &= \frac{\Delta}{\Sigma}\left\{ \frac{M(\Sigma - 2r^2)}{\Sigma^2}\dot{t}^2 + \frac{(r-M)\,\Sigma - r\Delta}{\Delta^2}\dot{r}^2 \right. \\
&+ \quad r\dot{\theta}^2 + \left[r + \left(\frac{\Sigma - 2r^2}{\Sigma^2} \right) a^2 M \sin^2\theta \right] \sin^2\theta\,\dot{\phi}^2 \\
&- \quad \left. 2aM\sin^2\theta \left(\frac{\Sigma - 2r^2}{\Sigma^2} \right) \dot{t}\dot{\phi} + \frac{a^2 \sin 2\theta}{\Delta}\dot{r}\dot{\theta} \right\} , \\
\ddot{\theta} &= \frac{1}{2\Sigma}\left(\sin 2\theta \left\{ \frac{2a^2 M r}{\Sigma^2}\dot{t}^2 - \frac{4aMr\,(r^2+a^2)^2}{\Sigma^2}\dot{t}\dot{\phi} - \frac{a^2}{\Delta}\dot{\phi}^2 \right. \right. \\
&+ \quad \left. \left. a^2\dot{\theta}^2 + \left[\Delta + \frac{2Mr\,(r^2+a^2)^2}{\Sigma^2} \right] \dot{\phi}^2 \right\} - 4r\dot{r}\dot{\theta} \right)
\end{aligned}
$$

Schwarzschild Geodesics



Kerr Geodesics

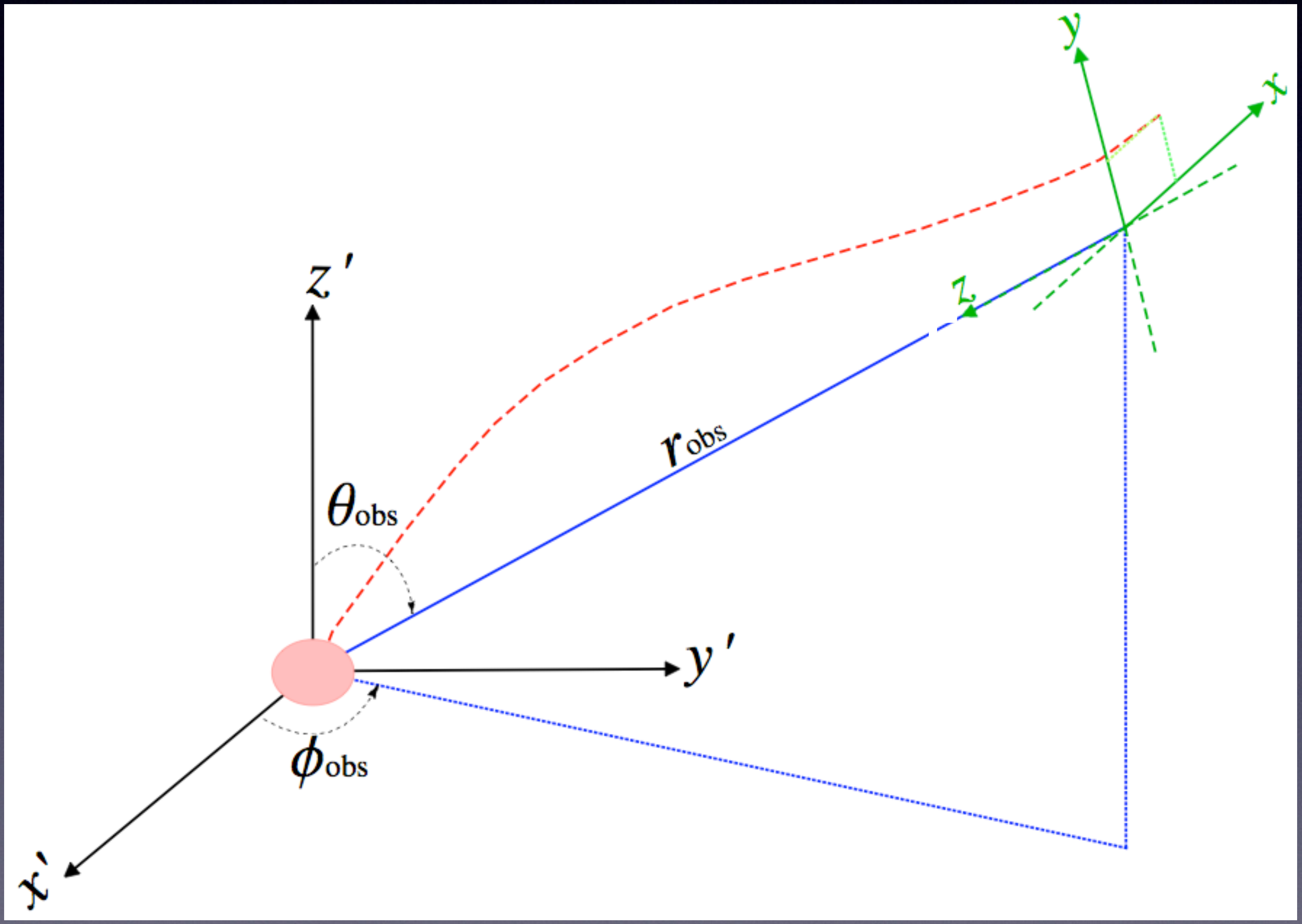
(a=0.998)



‘Seeing’ a Black Hole

- Although ‘invisible’, its presence is revealed through its interaction with nearby matter and radiation
- A black hole acts as a gravitational lens
- Radiation moving in its vicinity is not just deflected but also lensed due to the intense gravitational field
- To ‘see’ it, we must construct an observer grid and specify each photon by co-ordinates on this grid - each photon is now a pixel: integration is performed backwards in time
- To calculate an image we must specify for each ray the initial conditions $x^\alpha, \dot{x}^\alpha, E$ and L_z

Ray-Tracing Initialisation



- Observer grid represented by green axes
- z-axis of observer oriented towards black hole center
- x- and y-axes oriented as shown
- Black hole spin axis and z' axis taken to coincide
- Although ϕ_{obs} is arbitrary we keep it as a free parameter

Ray-Tracing Initialisation

- Calculate observer's co-ordinates in black hole co-ordinates:

$$\begin{aligned}\underline{\mathbf{x}}' &= \mathbf{A}_{y=x}\mathbf{R}_z(2\pi - \phi_{\rm obs})\mathbf{R}_x(\pi - \theta_{\rm obs})\underline{\mathbf{x}} + \mathbf{T}_{\mathbf{x}\to\mathbf{x}'} \\ &= \begin{pmatrix} \mathcal{D}(y,z)\cos\phi_{\rm obs} - x\sin\theta_{\rm obs} \\ \mathcal{D}(y,z)\sin\phi_{\rm obs} + x\cos\theta_{\rm obs} \\ (r_{\rm obs} - z)\cos\theta_{\rm obs} + y\sin\theta_{\rm obs} \end{pmatrix}\end{aligned}$$

$$\mathcal{D}(y,z) = \left(\sqrt{r_{\rm obs}^2 + a^2}\right)\sin\theta_{\rm obs} - y\cos\theta_{\rm obs}$$

- Determine initial velocity of the ray in black hole co-ordinates:

$$\underline{\dot{\mathbf{x}}}' = \begin{pmatrix} -\dot{x}\sin\phi_{\rm obs} - (\dot{y}\cos\theta_{\rm obs} + \dot{z}\sin\theta_{\rm obs})\cos\phi_{\rm obs} \\ \dot{x}\cos\phi_{\rm obs} - (\dot{y}\cos\theta_{\rm obs} + \dot{z}\sin\theta_{\rm obs})\sin\phi_{\rm obs} \\ \dot{y}\sin\theta_{\rm obs} - \dot{z}\cos\theta_{\rm obs} \end{pmatrix}$$

- We may then use the transformation between BL and Cartesian co-ordinates to calculate the I.C's $(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi})$ for the ray:

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\theta\cos\phi \\ y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

Ray-Tracing Initialisation

- The initial conditions of the ray may now be written as:

$$r = \sqrt{\frac{\sigma + \sqrt{\sigma^2 + 4a^2 z'^2}}{2}} \qquad \sigma \equiv x'^2 + y'^2 + z'^2 - a^2$$

$$\mathcal{R} \equiv \sqrt{r^2 + a^2}$$

$$\theta = \arccos\left(\frac{z'}{r}\right) \qquad \Phi \equiv \phi - \phi_{\rm obs}$$

$$\phi = \operatorname{atan2}(y', x')$$

$$\dot{r} = -\frac{r\mathcal{R}\sin\theta\sin\theta_{\rm obs}\cos\Phi + \mathcal{R}^2\cos\theta\cos\theta_{\rm obs}}{\Sigma}$$

$$\dot{\theta} = \frac{r\sin\theta\cos\theta_{\rm obs} - \mathcal{R}\cos\theta\sin\theta_{\rm obs}\cos\Phi}{\Sigma}$$

$$\dot{\phi} = \frac{\sin\theta_{\rm obs}\sin\Phi}{\mathcal{R}\sin\theta}$$

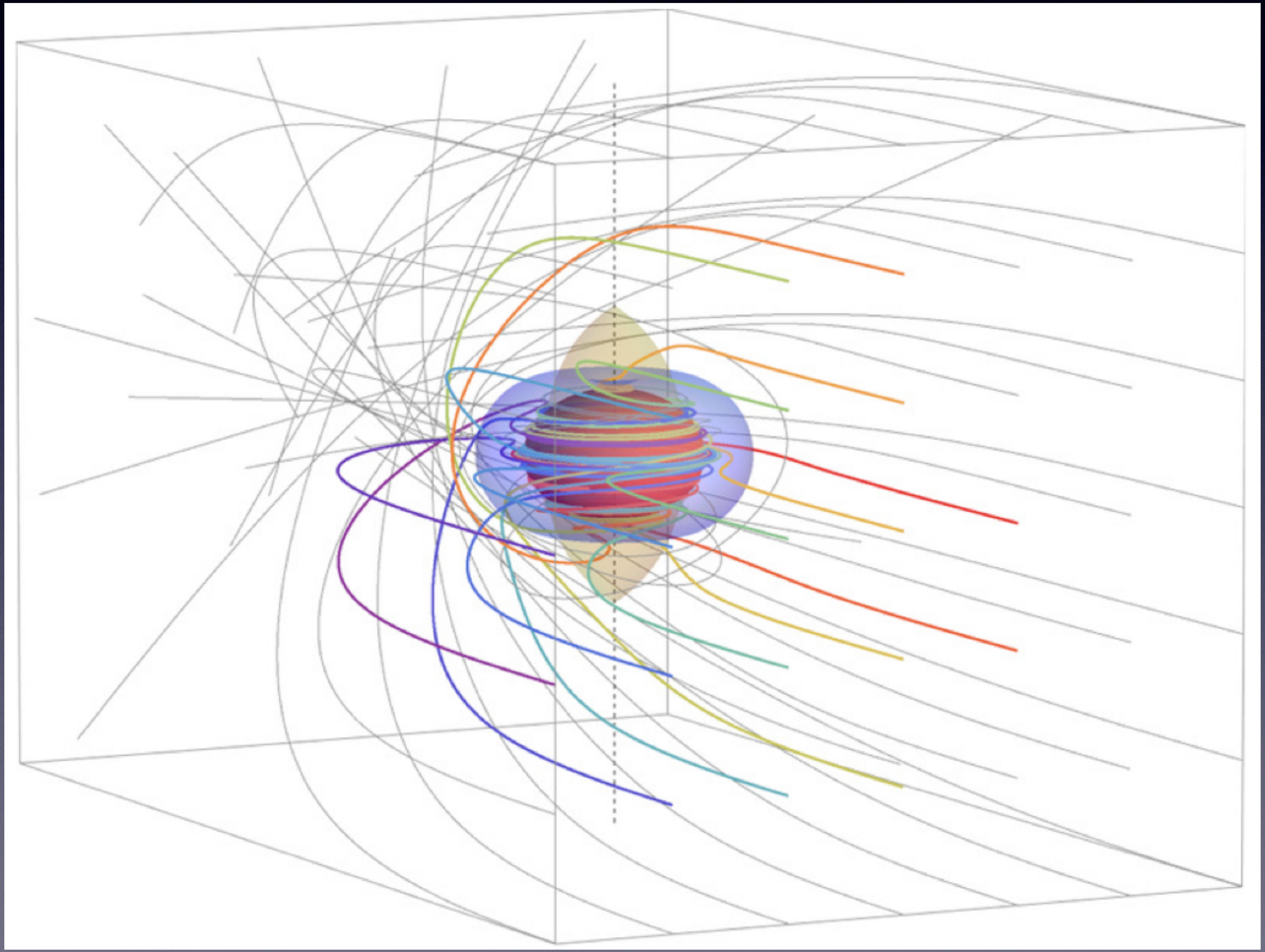
$$\dot{t} = \left[1 + \frac{2Mr(r^2 + a^2)}{\Sigma\Delta}\right]E - \left(\frac{2aMr}{\Sigma\Delta}\right)L_z$$

$$E^2 = \left(\frac{\Sigma - 2Mr}{\Sigma\Delta}\right)\left(\Sigma\,\dot{r}^2 + \Sigma\Delta\,\dot{\theta}^2 - \Delta\mu\right) + \Delta\sin^2\theta\,\dot{\phi}^2$$

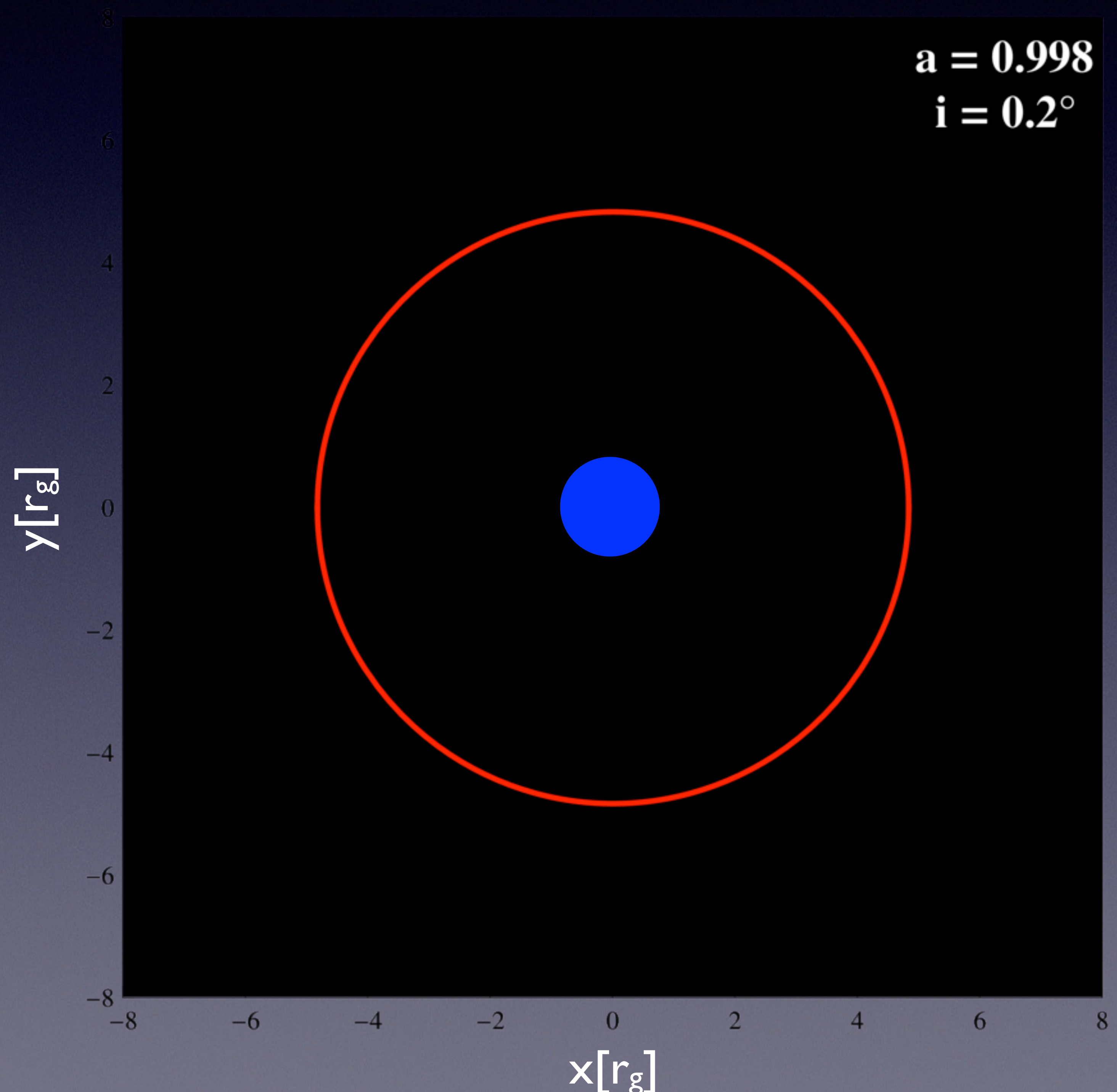
$$L_z = \frac{\left(\Sigma\Delta\dot{\phi} - 2aMrE\right)\sin^2\theta}{\Sigma - 2Mr}$$

- With the initial conditions $(t, \dot{t}, r, \dot{r}, \theta, \dot{\theta}, \phi, \dot{\phi}, E, L_z)$ we may now ray-trace an image
- In practical calculations we set *M* =1, which is equivalent to normalising the length scale to units of the gravitational radius

‘Seeing’ a Black Hole



Black Hole Shadow



(Classical) Radiative Transfer

Intensity

Absorption

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \quad \cancel{+ \iint \mathrm{d}\Omega' \, \mathrm{d}\nu' \, \sigma(\nu, \hat{\mathbf{\Omega}}; \nu', \hat{\mathbf{\Omega}}') I_{\nu'}(\hat{\mathbf{\Omega}}')}$$

Scattering

Path Length

Emission

$$\tau_\nu(s) = \int_{s_0}^{s} \mathrm{d}s' \, \alpha_\nu(s')$$

Optical Depth

$$S_\nu = \frac{j_\nu}{\alpha_\nu}$$

Source Function

$$\begin{aligned} I_\nu(s) &= I_\nu(s_0)\, \mathrm{e}^{-\tau_\nu} + \int_{s_0}^{s} \mathrm{d}s' \, j_\nu(s') \, \mathrm{e}^{-[\tau_\nu(s) - \tau_\nu(s')]} \\ &= I_\nu(s_0)\, \mathrm{e}^{-\tau_\nu} + \int_{0}^{\tau_\nu} \mathrm{d}\tau'_\nu \, S_\nu(\tau'_\nu) \, \mathrm{e}^{-(\tau_\nu - \tau'_\nu)} \end{aligned}$$

Covariant Radiative Transfer

- Consider a bundle of particles threading a phase space volume defined as $\mathrm{d}\mathcal{V} = \mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{p}$
- Two important conserved quantities result:

  (1) *conservation of particle number in the bundle*

  (2) *conservation of phase space volume, i.e.* $\dfrac{\mathrm{d}\mathcal{V}}{\mathrm{d}\lambda} = 0$ ← *affine parameter*

- These two conserved quantities imply an invariant quantity:

$$f\left(x^i, p^i\right) = \frac{\mathrm{d}N}{\mathrm{d}\mathcal{V}}$$

Covariant Radiative Transfer

- For relativistic particles:

$$|\vec{p}| = E$$

$$\mathrm{d}^3\vec{p} = E^2 \mathrm{d}E\mathrm{d}\Omega$$

$$\mathrm{d}^3\vec{x} = \mathrm{d}A\mathrm{d}t$$

$$\longrightarrow f\left(x^i, p^i\right) = \frac{\mathrm{d}N}{E^2\mathrm{d}A\mathrm{d}t\mathrm{d}E\mathrm{d}\Omega}$$

- The specific intensity of a ray is given by:

$$I_E = \frac{E\mathrm{d}N}{\mathrm{d}A\mathrm{d}t\mathrm{d}E\mathrm{d}\Omega} \longrightarrow \mathcal{I} \equiv \frac{I_E}{E^3} = \frac{I_\nu}{\nu^3}$$

Lorentz invariant intensity

Covariant Radiative Transfer

- The velocity of a particle in the co-moving frame of a medium is:

$$v^\beta = \left(g^{\alpha\beta} + u^\alpha u^\beta\right) k_\alpha$$

- The variation in path length w.r.t. affine parameter is given by:

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}\lambda} = -\left\|v^\beta\right\|\Big|_{\lambda_{\mathrm{obs}}} &= -\sqrt{k_\beta k^\beta + \left(k_\alpha u^\alpha\right)^2\left(u_\beta u^\beta + 2\right)} \\ &= -k_\alpha u^\alpha\Big|_{\lambda_{\mathrm{obs}}}\end{aligned}$$

- The energy shift is:

$$\gamma^{-1} = \frac{\nu_0}{\nu} = \frac{-k_\alpha u^\alpha\big|_\lambda}{E_{\mathrm{obs}}} = \frac{k_\alpha u^\alpha\big|_\lambda}{k_\beta u^\beta\big|_{\lambda_{\mathrm{obs}}}}$$

Covariant Radiative Transfer

- Optical depth, τ, is an invariant quantity
- Lorentz invariant absorption coefficient: $\chi = \nu\alpha_\nu$
- Lorentz invariant emission coefficient: $\eta = \nu^{-2} j_\nu$
- We may now write down the Lorentz invariant RT equation as:

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\tau_\nu} = -\mathcal{I} + \frac{\eta}{\chi} = -\mathcal{I} + \mathcal{S}$$

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}s} = -\alpha_\nu \mathcal{I} + \frac{j_\nu}{\nu^3}$$

$$\alpha_{0,\nu} \equiv \alpha_0\left(x^\beta, \nu\right)$$
$$j_{0,\nu} \equiv j_0\left(x^\beta, \nu\right)$$

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\lambda} = -k_\alpha u^\alpha\Big|_\lambda \left(\alpha_{0,\nu}\mathcal{I} + \frac{j_{0,\nu}}{\nu^3}\right)$$

General-Relativistic Radiative Transfer

- We may solve the GRRT equation and obtain the intensity as:

$$\mathcal{I}(\lambda) = \mathcal{I}(\lambda_0)\mathrm{e}^{-\tau_\nu(\lambda)} - \int_{\lambda_0}^{\lambda} \mathrm{d}\lambda'' \, \frac{j_{0,\nu}(\lambda'')}{\nu_0^3} \exp\left(- \int_{\lambda''}^{\lambda} \mathrm{d}\lambda' \, \alpha_{0,\nu}(\lambda') k_\alpha u^\alpha |_{\lambda'} \right) k_\alpha u^\alpha |_{\lambda''}$$

where the optical depth is defined as:

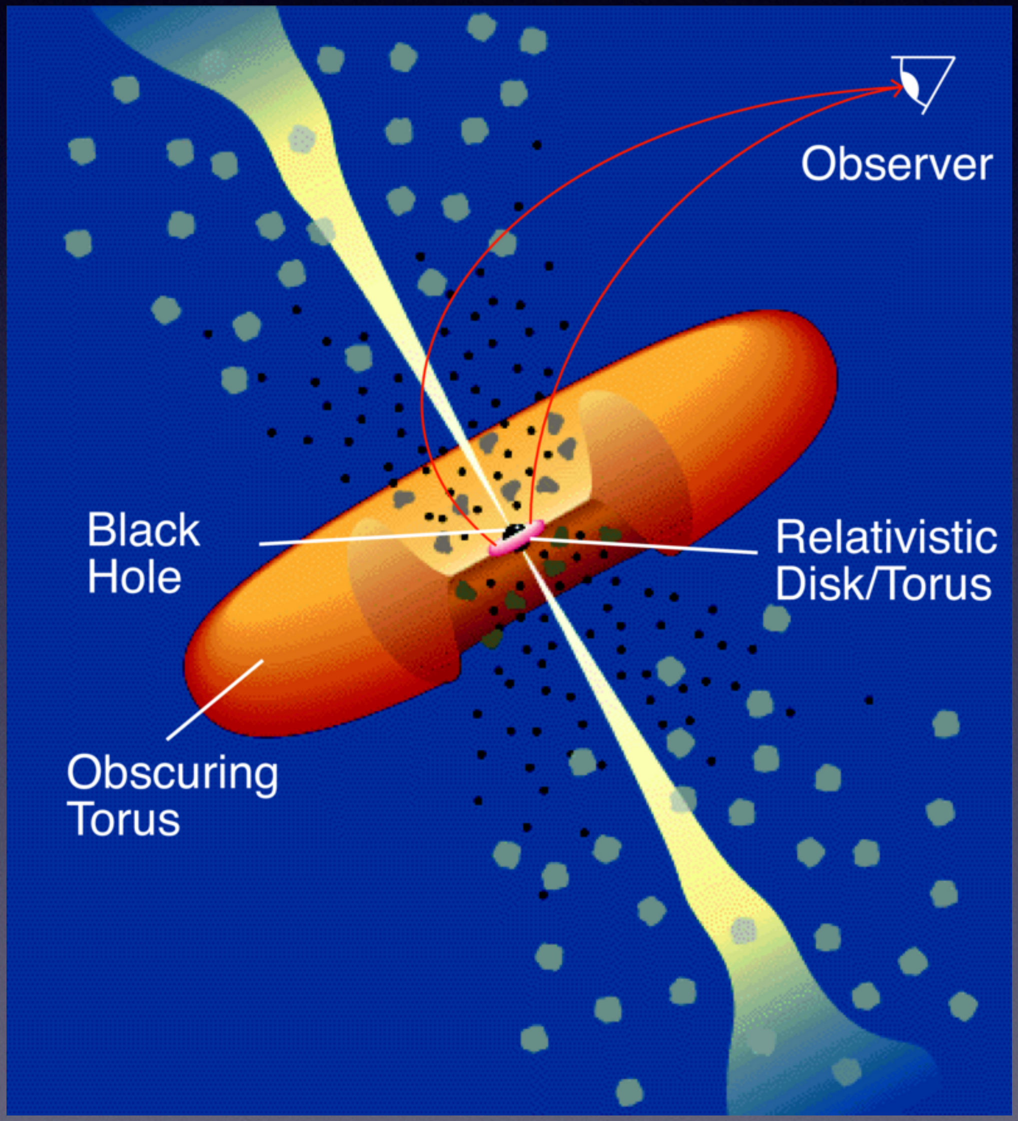
$$\tau_\nu(\lambda) = - \int_{\lambda_0}^{\lambda} \mathrm{d}\lambda' \, \alpha_{0,\nu}(\lambda') k_\alpha u^\alpha |_{\lambda'}$$

- We may now decouple the GRRT equation into two ODEs:

$$\frac{\mathrm{d}\tau_\nu}{\mathrm{d}\lambda} = \gamma^{-1}\alpha_{0,\nu} \, ,$$

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\lambda} = \gamma^{-1}\left(\frac{j_{0,\nu}}{\nu_0^3}\right)\mathrm{e}^{-\tau_\nu}$$

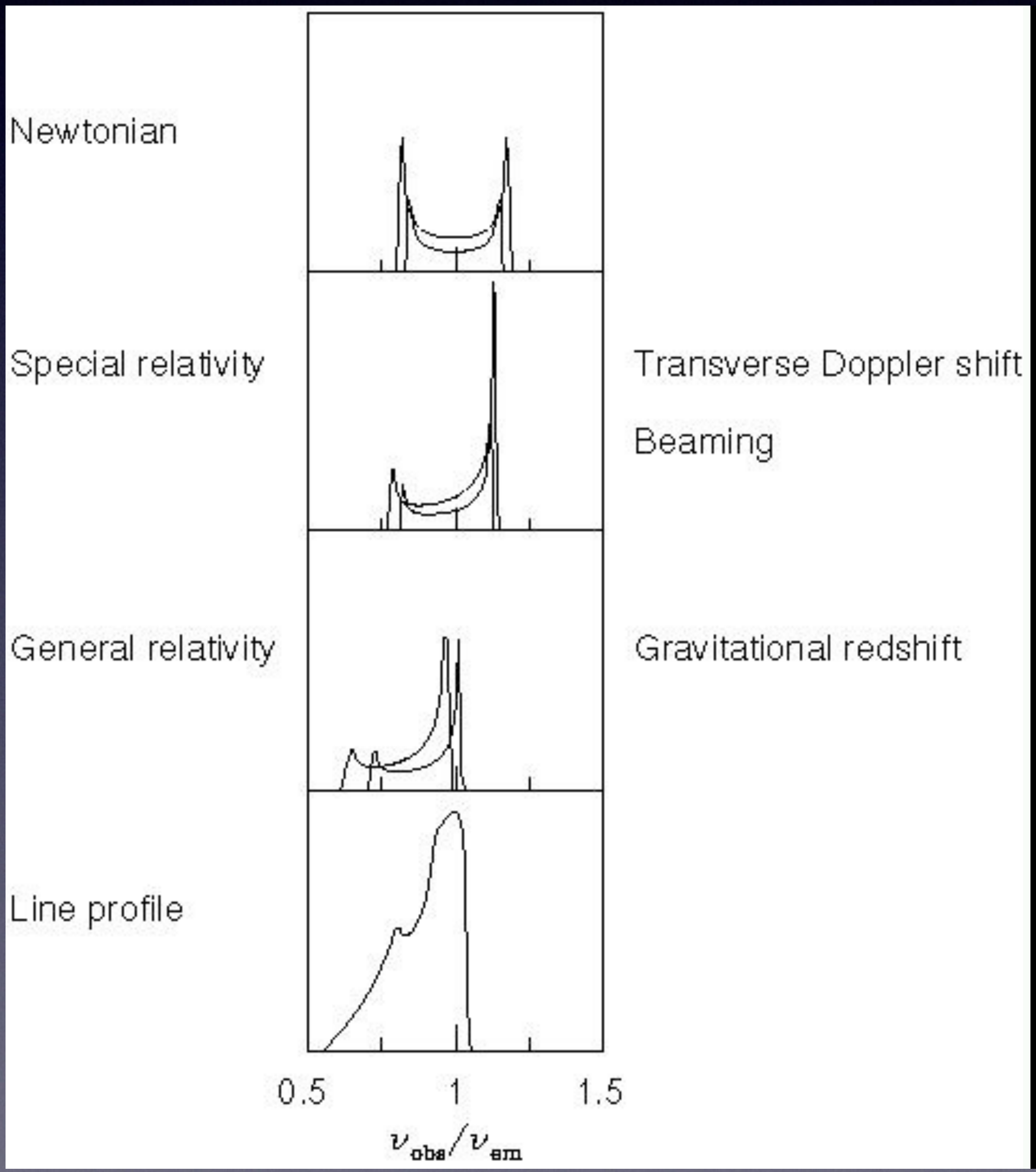
GR Radiative Transfer

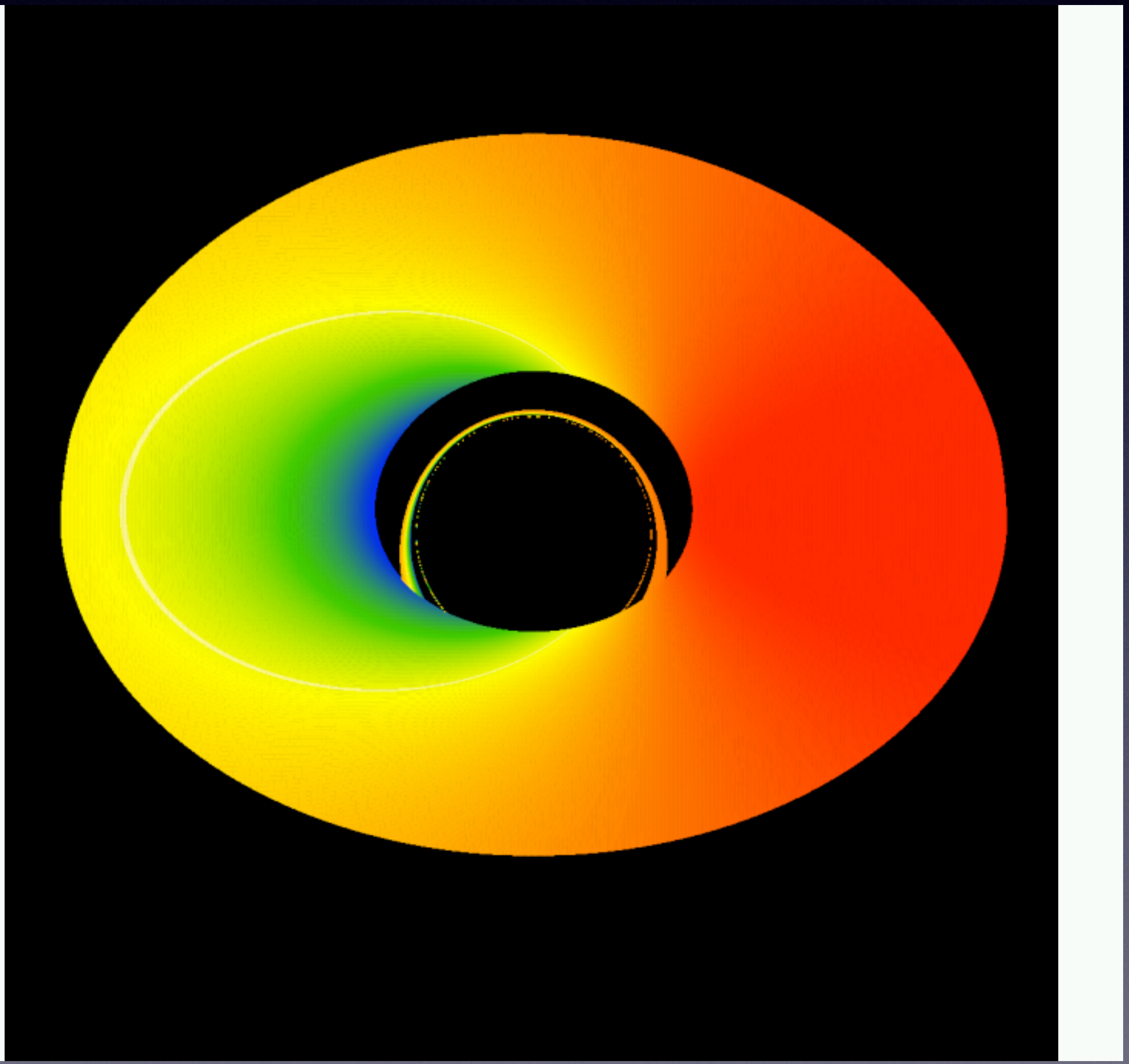


Adapted from C.M. Urry and P. Padovani

- Specify space time metric
- Solve photon geodesics
- Solve RTE along geodesics
- Assume as a first test a geometrically thin, optically thick disk (Shakura & Sunyaev 1973)
- Disk scale height negligible compared to its radial extent, effectively 2D

The Formation of an Emission Line

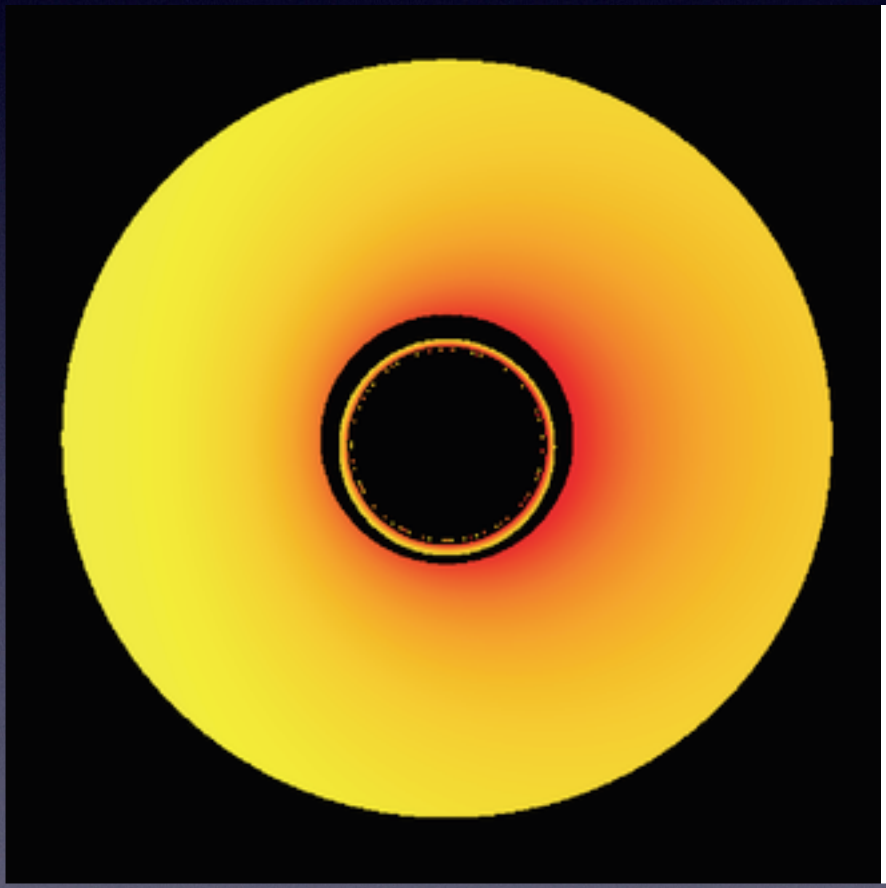




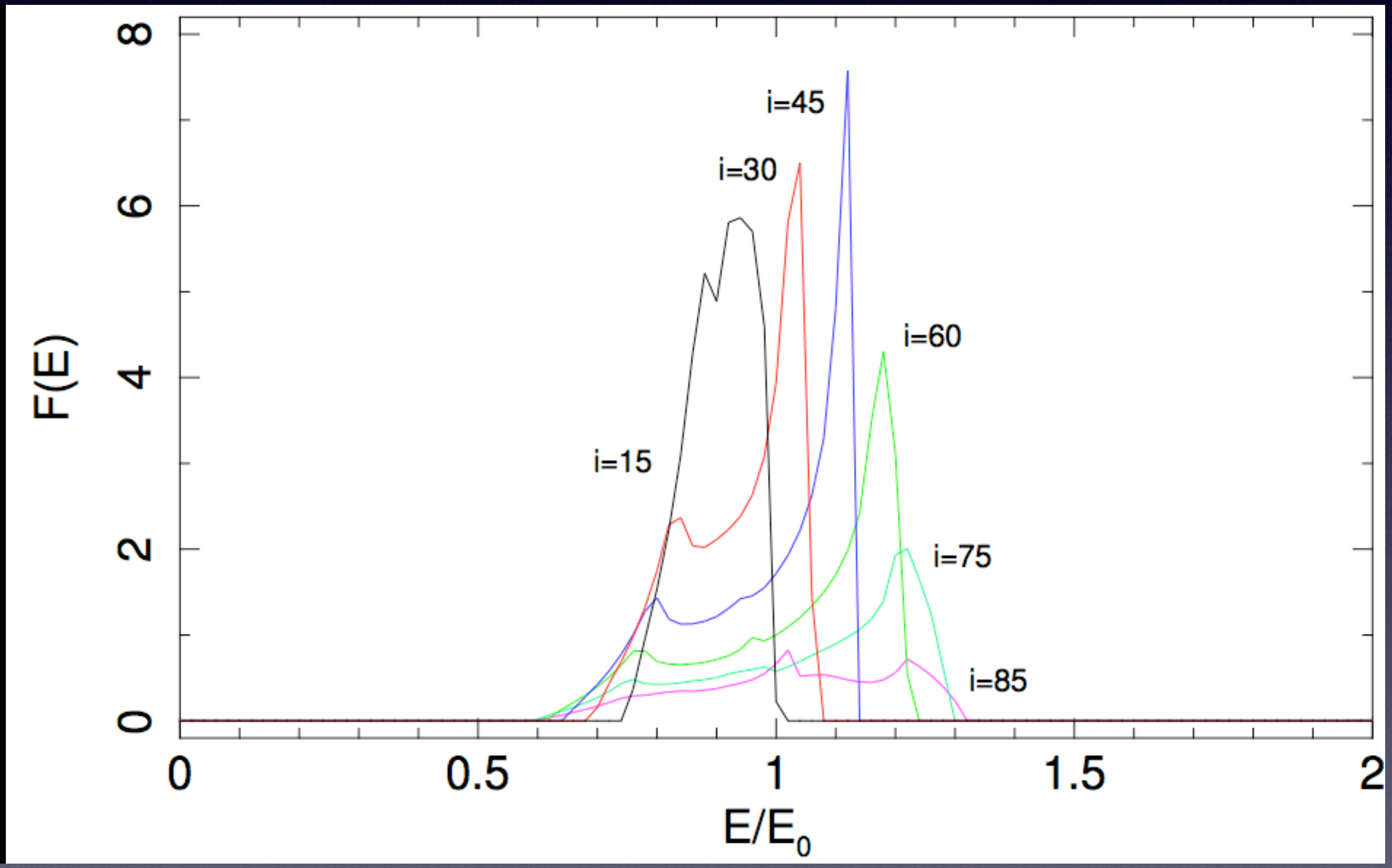
Fabian et al. 2000

Tanaka et al. 1995, Nandra et al. 1997

Optically Thick Accretion Disk



Energy shift



Emission line profile

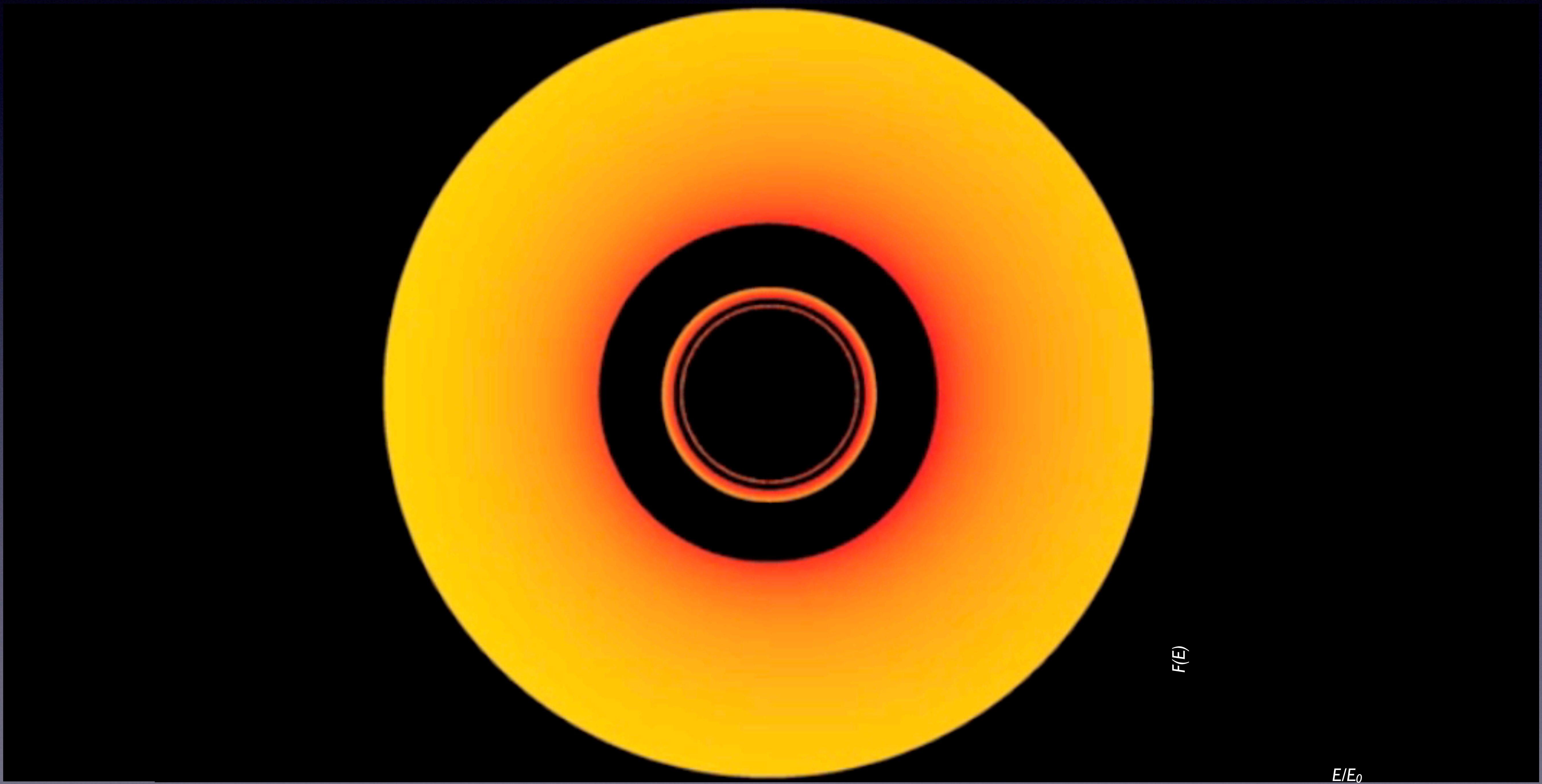
Optically Thick Accretion Tori

- Assume optical depth $\tau >> 1$
- Torus is stationary, axisymmetric and rotationally supported
- Internal structure irrelevant
- Solve torus equations of motion to determine parametric equations describing emission boundary surface
- Specify angular velocity profile for torus:

$$\Omega(r\sin\theta) = \frac{\sqrt{M}}{(r\sin\theta)^{3/2} + a\sqrt{M}} \left(\frac{r_{\mathrm{k}}}{r\sin\theta}\right)^n$$

- Torus is supported by pressure forces arising from the differential rotation of neighboring fluid elements

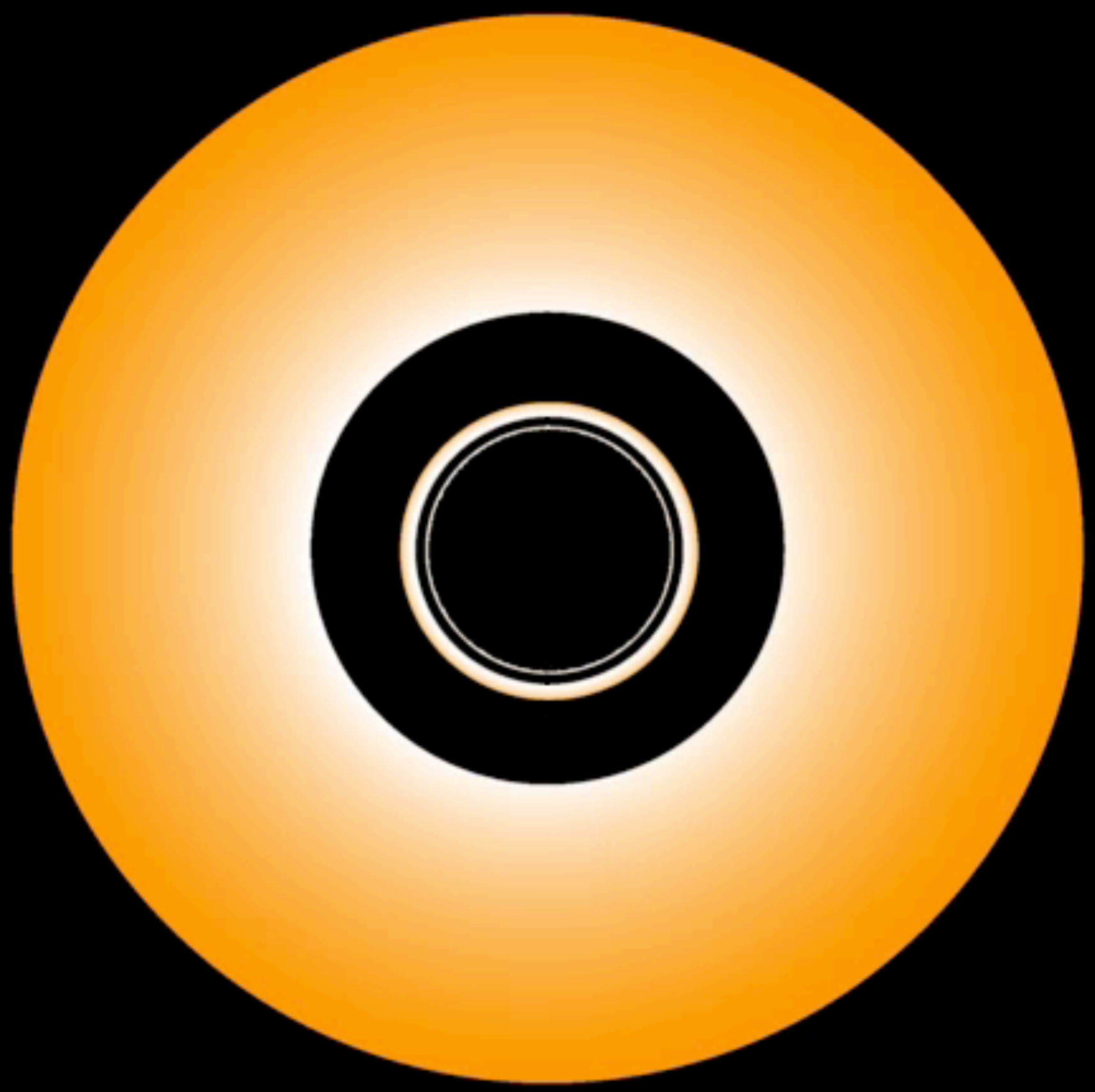
Optically Thick Accretion Torus

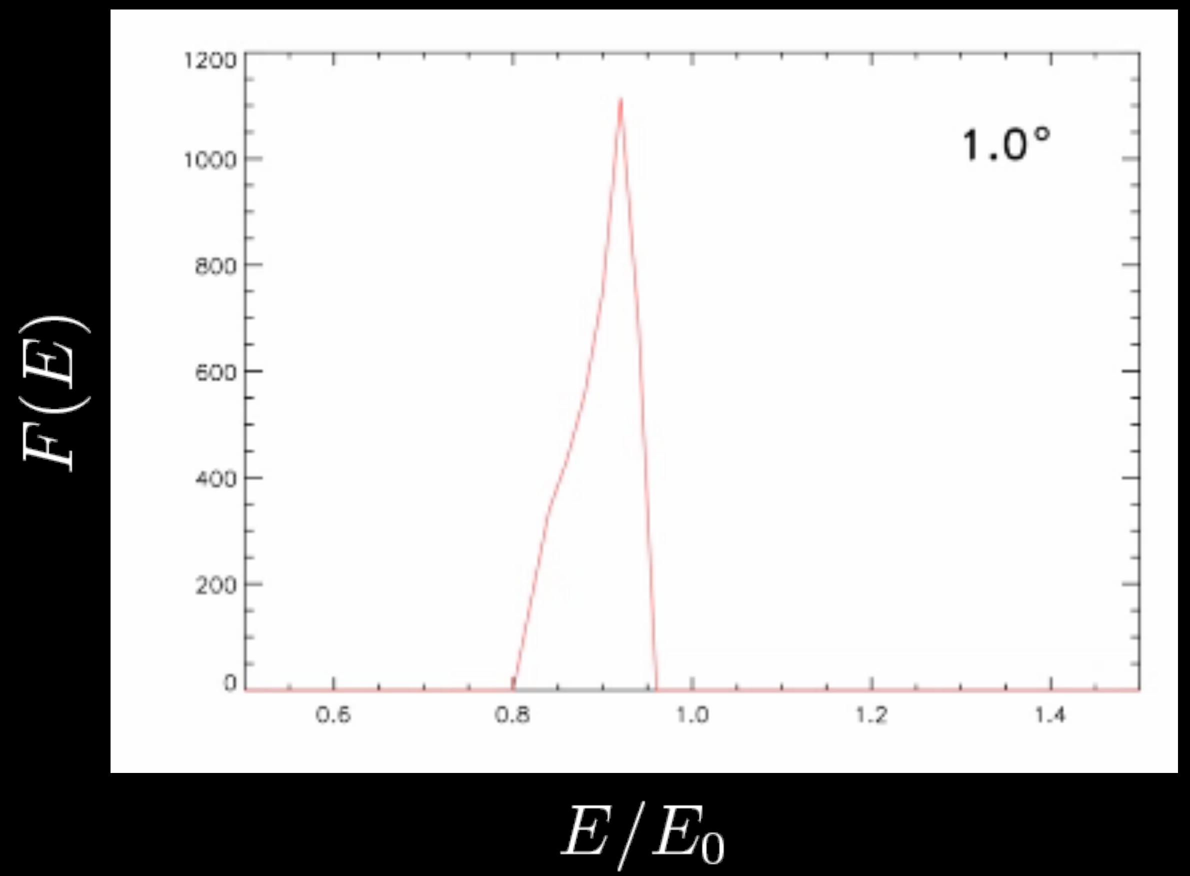


Energy shift

Emission line spectrum

Optically Thick Accretion Torus





Intensity

Emission line spectrum

Optically Thin Accretion Tori

- Construct a general relativistic perfect fluid:

$$T^{\alpha\beta} = (\rho + P + \epsilon)u^\alpha u^\beta + P g^{\alpha\beta}$$

- The momentum equation yields, for a static, axisymmetric configuration:

$$\frac{\partial_\alpha P}{\rho + P + \epsilon} = \partial_\alpha \ln(u_t) - \frac{\Omega \partial_\alpha l}{1 - l\Omega}$$

- Total pressure within torus is the sum of the gas and radiation pressures:

$$P_{\rm gas} = \frac{\rho k_{\rm B} T}{\mu m_{\rm H}} = \beta P$$

$$P_{\rm rad} = \frac{4\sigma}{3} T^4 = (1-\beta) P$$

$$P = \hbar \left[\frac{45(1-\beta)}{\pi^2 (\mu m_{\rm H} \beta)^4}\right]^{1/3} \rho^{4/3}$$

Optically Thin Accretion Tori

- Assume a polytropic equation of state for the fluid within the torus to close the system of equations for pressure:

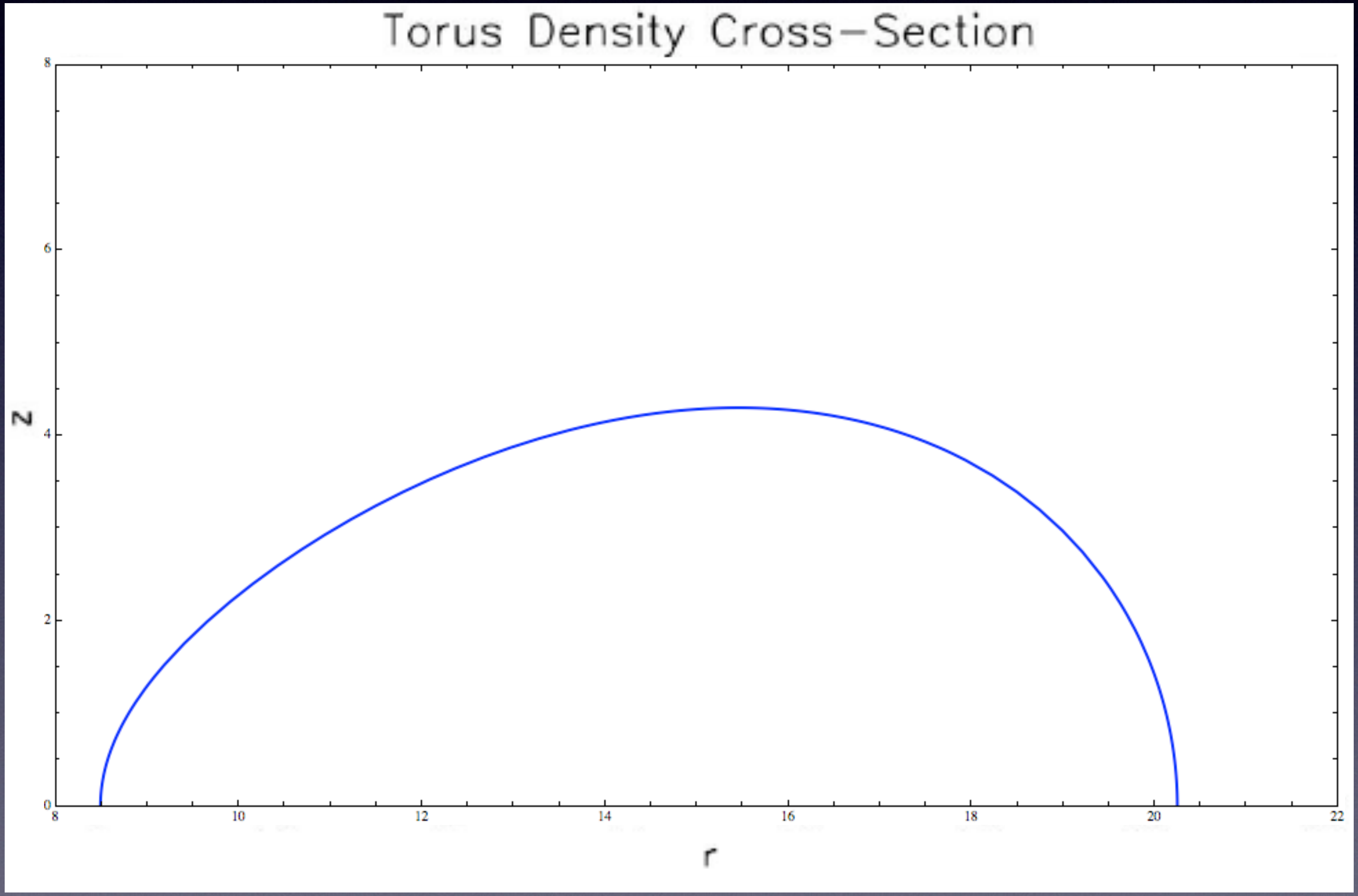
$$P = \kappa\rho^{\Gamma}$$

- Inserting this into the fluid equations yields the torus density structure:

$$\boxed{\partial_\alpha \rho = \partial_\alpha \xi \left(\frac{\rho^{2-\Gamma}}{\kappa\Gamma} + \frac{\rho}{\Gamma - 1} \right)}$$

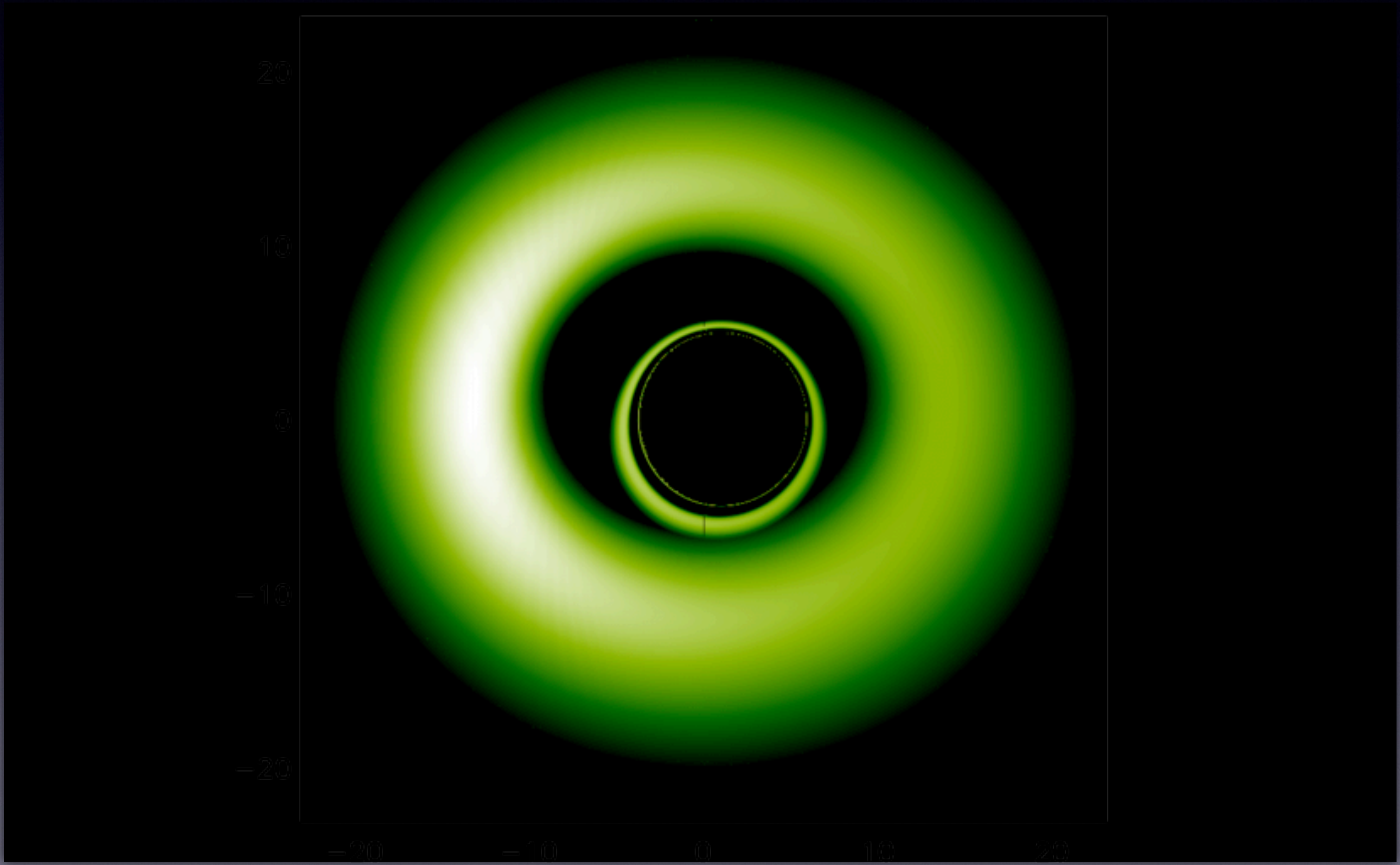
Define a new variable: $\xi = \ln(\Gamma - 1 + \Gamma\kappa\rho^{\Gamma-1})$

Optically Thin Accretion Tori



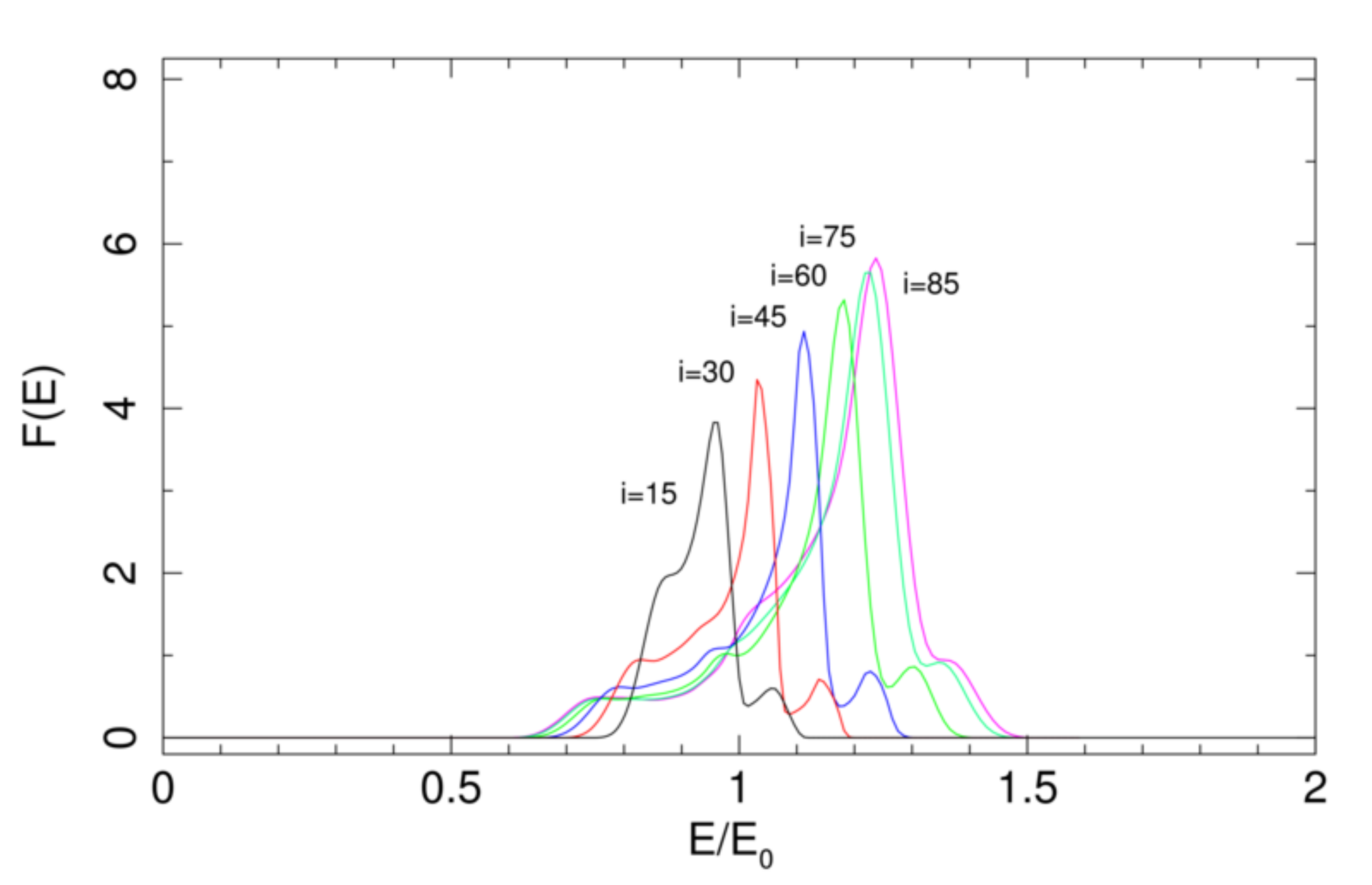
$$
\begin{aligned}
n &= 0.21 \\
r_{\mathrm{k}} &= 12 r_{\mathrm{g}} \\
\beta &= 1.235 \times 10^{-5} \\
\rho_{\mathrm{c}} &= 10^{11} \mathrm{cm}^{-3} \\
a &= 0.998
\end{aligned}
$$

Emission From Optically Thin Accretion Torus



Intensity

Emission From Optically Thin Accretion Torus



Multiple (blended) emission lines from an optically thin accretion torus

Emission From Quasi-Opaque Accretion Torus

- Consider two opacity sources with emission and corresponding absorption coefficients in the rest frame given by:

$$j_{0,1}(E_0) = \mathcal{K}\left(\frac{n_e}{\mathrm{cm}^{-3}}\right)^2\left(\frac{E_0}{\mathrm{keV}}\right)^{-1}\left(\frac{k_B T}{\mathrm{keV}}\right)^{-1/2} e^{-E_0/k_B T},$$

$$j_{0,2}(E_0) = \mathcal{C}\left(\frac{n_e}{\mathrm{cm}^{-3}}\right)\left(\frac{E_0}{\mathrm{keV}}\right)^{-2.5},$$

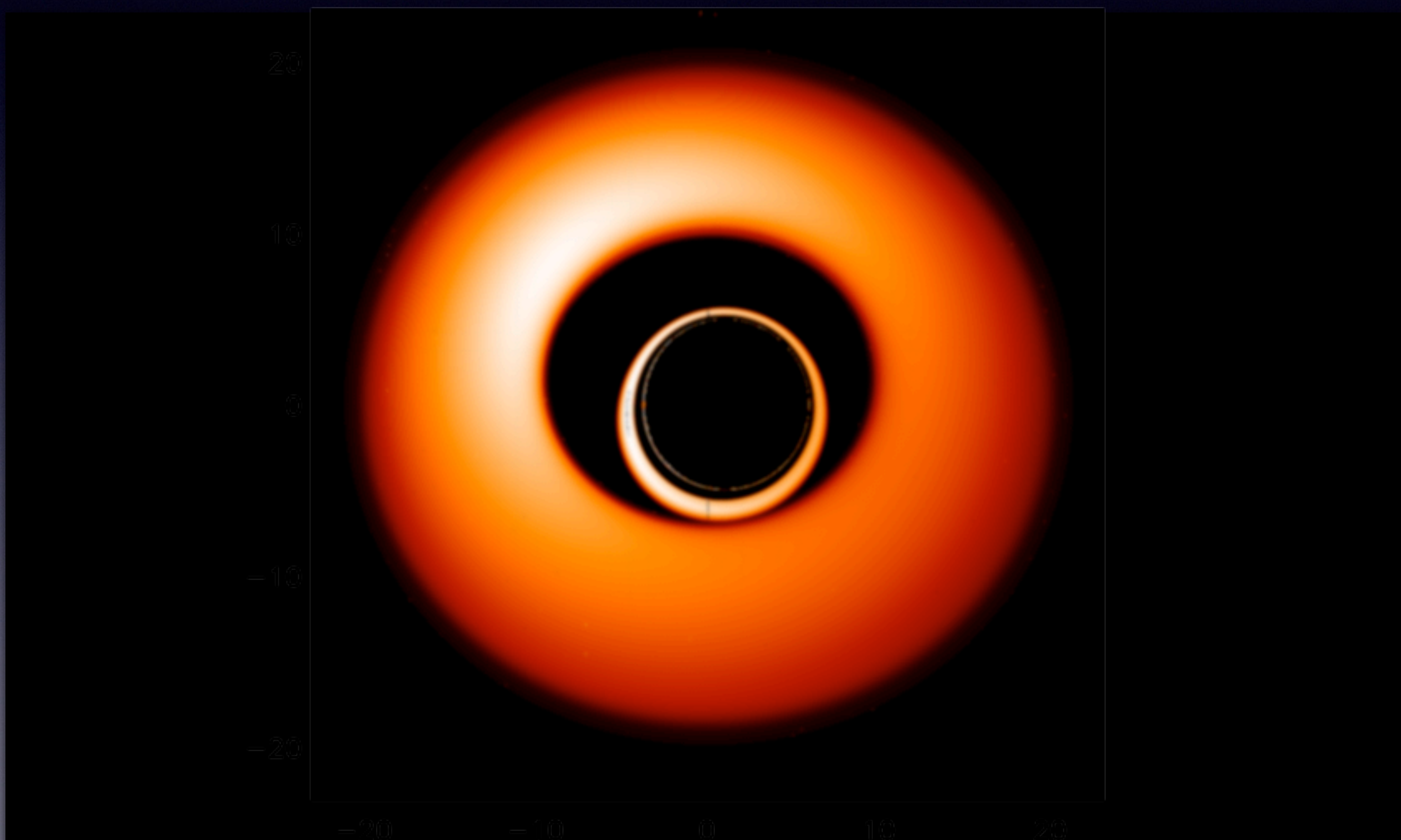
$$\alpha_{0,1}(E_0) = B_1\left(\frac{n_e}{\mathrm{cm}^{-3}}\right)^2 \sigma_T f_1(E_0)\,\mathrm{cm}^{-1},$$

$$\alpha_{0,2}(E_0) = B_2\left(\frac{n_e}{\mathrm{cm}^{-3}}\right) \sigma_T f_2(E_0)\mathrm{cm}^{-1}.$$

$$\mathcal{K} = 8\times 10^{-46}\,\mathrm{erg\,s^{-1}\,cm^{-3}\,Hz^{-1}}$$
$$\mathcal{C} = 2.162\times 10^{-45}\,\mathrm{erg\,s^{-1}\,cm^{-3}\,Hz^{-1}}$$
$$B_1 = 0$$

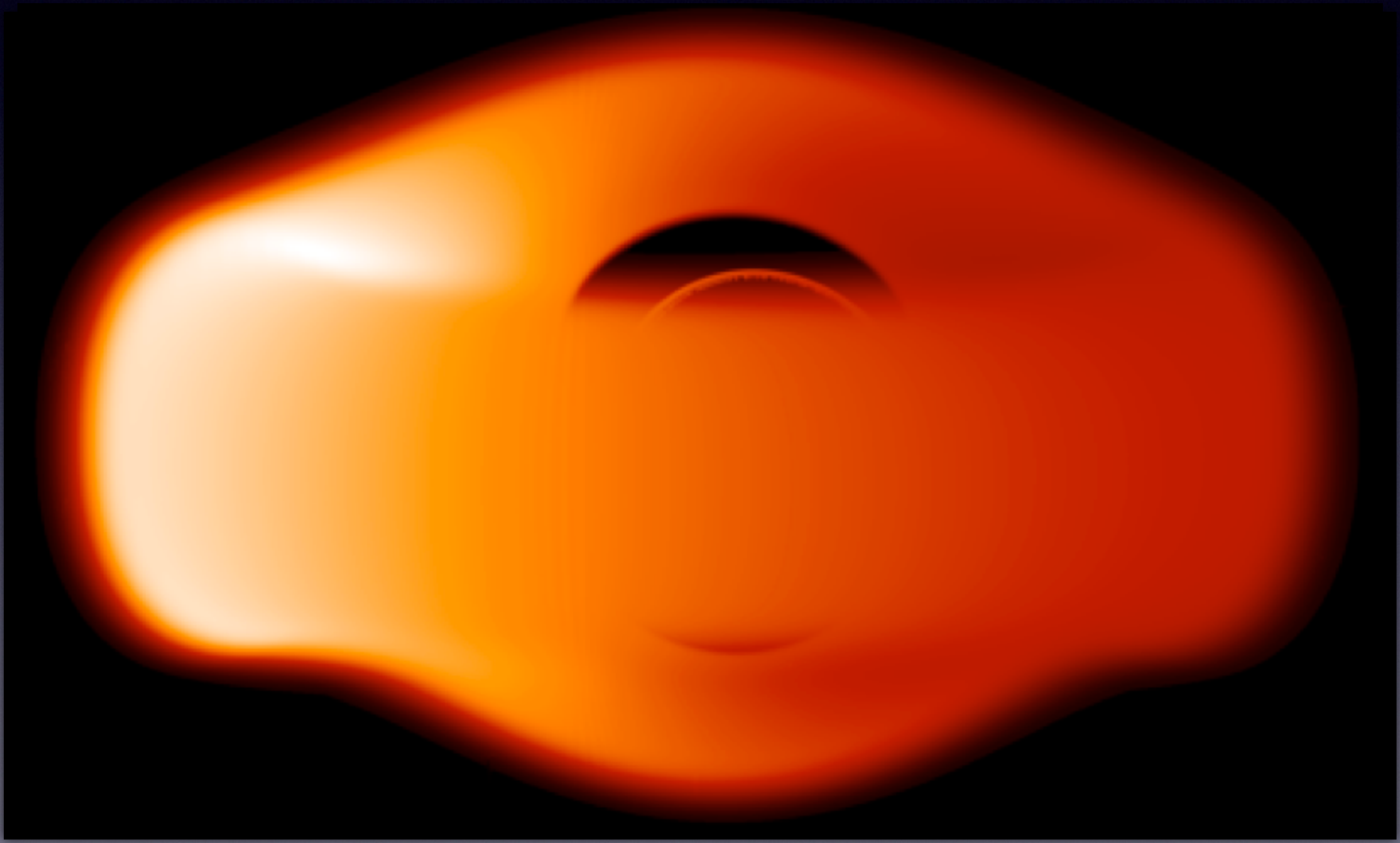
B_2 is chosen such that $\alpha_0 r_{out}$=1-5 across the torus

Emission From Quasi-Opaque Accretion Torus



Intensity

Emission From Quasi-Opaque Accretion Torus



Intensity

GR Compton Scattering

- When scattering is included the RTE takes the form:

$$\frac{d\mathcal{I}\left(x^{\beta},k^{\beta}\right)}{d\lambda} = -k_{\alpha}u^{\alpha}|_{\lambda}\left[\eta_0\left(x^{\beta},k^{\beta}\right) - \chi_0\left(x^{\beta},k^{\beta}\right)\mathcal{I}\left(x^{\beta},k^{\beta}\right) + \int d^4k'^{\beta}\,\sigma\left(x^{\beta};k^{\beta},k'^{\beta}\right)\mathcal{I}\left(x^{\beta},k'^{\beta}\right)\right]$$

- Solving the above integro-differential equation is analytically impossible except in very symmetrical, idealised situations
- A covariant form of the Eddington approximation (e.g. Thorne 1981, Fuerst & Wu 2006) is needed to reduce the problem to solving a system of coupled ODEs
- No available codes to do this - reliant on Monte-Carlo simulations and semi-analytic approaches that are restrictive

GR Compton Scattering

- The scattering kernel and its angular moments must be evaluated covariantly
- First the Compton scattering cross-section must be rewritten:

$$\boxed{\sigma\left(\gamma \to \gamma', n^{\alpha} \to n'^{\alpha}, v^{\alpha}\right) = \frac{3\sigma_{\mathrm{T}}}{16\pi\gamma\nu\lambda}\left[1 + \left(1 + \frac{m_e^2 \mathcal{T}}{k^{\alpha}k'_{\alpha}}\right)^2 + \mathcal{T}\right]\delta\left(\frac{\mathcal{P}}{m_{\mathrm{e}^2\gamma\gamma'}}\right)}$$

$$\mathcal{T} = \frac{\left(k^{\alpha}k'_{\alpha}\right)^2}{(p^{\alpha}k_{\alpha})(p^{\beta}p'_{\beta})}$$

$$\mathcal{P} = k^{\alpha}k'_{\alpha} + p^{\alpha}k'_{\alpha} - p^{\alpha}k_{\alpha}$$

GR Compton Scattering

- After some mathematical tricks and physical insight, angular moments of the scattering kernel may be written in the following symmetrical form:

$$\int d\zeta\, \zeta^n \sigma_{\mathrm{S}}\left(\gamma \to \gamma', \zeta, \tau\right) = \frac{3\rho\sigma_{\mathrm{T}}}{8\gamma\nu} \int_{-1}^{1} d\zeta\, \zeta^n \int_{\lambda_+}^{\infty} d\lambda\, \frac{f(\lambda)}{\lambda^5} \left[\frac{2\gamma\gamma'}{q} + R(\gamma+\lambda) - R(\gamma'-\lambda)\right]$$

$$R(x) = \frac{x(\gamma^{-1}+\gamma'^{-1}) - (1+\zeta)}{(1-\zeta)^2(x^2+\omega^2)^{3/2}} + \left[-\gamma\gamma' + \frac{2}{1-\zeta} + \frac{2}{\gamma\gamma'(1-\zeta)^2}\right] \frac{1}{(x^2+\omega^2)^{1/2}}$$

$$\omega^2 \equiv \frac{1+\zeta}{1-\zeta}$$

- The next step is to perform the above integrals

GR Compton Scattering

- First change the order of integration:

$$\int_{-1}^{1} d\zeta \int_{\lambda_+}^{\infty} d\lambda = \int_{\lambda_L}^{\infty} d\lambda \int_{-1}^{\zeta_+} d\zeta + \int_{\lambda_{\min}}^{\lambda_L} d\lambda \int_{\zeta_-}^{\zeta_+} d\zeta$$

- Next define three angular moment integrals:

$$Q_n = \int d\zeta \frac{\zeta^n}{q}$$

$$R_n = \int \frac{d\zeta\, \zeta^n}{(1-\zeta)^2 \left(x^2 + \frac{1+\zeta}{1-\zeta}\right)^{3/2}}$$

$$S_{n,m} = \int \frac{d\zeta\, \zeta^n}{(1-\zeta)^m \left(x^2 + \frac{1+\zeta}{1-\zeta}\right)^{1/2}}$$

GR Compton Scattering

- Introduce the Gauss Hypergeometric function:

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!} z^n $$

$$ (a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)} $$

- This series is absolutely convergent for $|z| < 1$
- In all of our calculations $z \in \mathbb{R}$ and $z < 1$
- The case $z \leq -1$ may be solved by analytic extension:

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a,b;c;\frac{z}{z-1}\right) $$

GR Compton Scattering

- With the aforementioned hypergeometric function we may now write the moment integrals in closed form:

$$
\begin{aligned}
Q_n &= \frac{\zeta^{n+1}}{(n+1)\sqrt{\gamma^2+\gamma'^2}}\mathcal{F}\left(\frac{1}{2}, n+1, x_1\right) \\
R_n &= -\frac{(1-\zeta)^{1/2}}{\sqrt{2}}\sum_{k=0}^{n}\frac{{}_nC_k(\zeta-1)^k}{2k+1}\mathcal{F}\left(\frac{3}{2}, k+\frac{1}{2}, x_2\right) \\
S_{n,m} &= -\frac{(1-\zeta)^{\frac{3}{2}-m}}{\sqrt{2}}\sum_{k=0}^{n}\frac{{}_nC_k(\zeta-1)^k}{k-m+\frac{3}{2}}\mathcal{F}\left(\frac{1}{2}, k-m+\frac{3}{2}, x_2\right)
\end{aligned}
$$

$$\mathcal{F}(a_1, a_2, x) = {}_2F_1(a_1, a_2; a_2+1; x)$$

$$
\begin{aligned}
x_1 &\equiv \frac{2\gamma\gamma'}{\gamma^2+\gamma'^2}\zeta \\
x_2 &= \frac{1}{2}(1-x^2)(1-\zeta)
\end{aligned}
$$

- There already exist numerical codes to evaluate ${}_2F_1$ accurately

GR Compton Scattering

- We may now write the angular moments of the Compton scattering kernel as:

$$\sigma_n(\gamma \to \gamma', \tau) = \frac{\mathcal{C}\mathrm{e}^{-1/\tau}}{\gamma^2 \tau \mathrm{K}_2(1/\tau)} T_n(\gamma, \gamma', \tau)$$

$$\mathcal{C} = \frac{3\rho\sigma_{\mathrm{T}}}{32\pi m_{\mathrm{e}}}$$

$$\begin{aligned} T_n &= T_{1,n} + T_{2,n} \\ T_{1,n} &= t_{1,n} + t_{3,n} - t_{4,n} - t_{5,n} + t_{6,n} \\ T_{2,n} &= t_{2,n} + t_{7,n} - t_{8,n} - t_{9,n} + t_{10,n} \end{aligned}$$

$$\begin{aligned} t_{1,n} &= I_{1,n}(\zeta_+, \lambda_{\mathrm{L}}, \infty) - 2\gamma\gamma'\tau Q_n(-1)\mathrm{e}^{-\lambda_{\mathrm{L}}/\tau} \\ t_{2,n} &= I_{1,n}(\zeta_+, \lambda_{\min}, \lambda_{\mathrm{L}}) - I_{1,n}(\zeta_-, \lambda_{\min}, \lambda_{\mathrm{L}}) \\ t_{3,n} &= f(\zeta_+, \gamma + \lambda, \lambda_{\mathrm{L}}, \infty) \\ t_{4,n} &= f(-1, \gamma + \lambda, \lambda_{\mathrm{L}}, \infty) \\ t_{5,n} &= g(\zeta_+, \gamma' - \lambda, \lambda_{\mathrm{L}}, \infty) \end{aligned}$$

$$\begin{aligned} t_{6,n} &= g(-1, \gamma' - \lambda, \lambda_{\mathrm{L}}, \infty) \\ t_{7,n} &= f(\zeta_+, \gamma + \lambda, \lambda_{\min}, \lambda_{\mathrm{L}}) \\ t_{8,n} &= f(\zeta_-, \gamma + \lambda, \lambda_{\min}, \lambda_{\mathrm{L}}) \\ t_{9,n} &= g(\zeta_+, \gamma' - \lambda, \lambda_{\min}, \lambda_{\mathrm{L}}) \\ t_{10,n} &= g(\zeta_-, \gamma' - \lambda, \lambda_{\min}, \lambda_{\mathrm{L}}) \end{aligned}$$

GR Compton Scattering

- The additional terms are defined as:

$$f(\zeta, x, \lambda_1, \lambda_2) = \frac{\gamma}{\gamma'} I_{2,n} + I_{3,n} - I_{4,n} - I_{5,n} + I_{6,n} + I_{7,n}$$

$$g(\zeta, x, \lambda_1, \lambda_2) = f(\zeta, x, \lambda_1, \lambda_2) - \left[\left(\frac{\gamma}{\gamma'} - \frac{\gamma'}{\gamma}\right) I_{2,n} + 2 I_{3,n}\right]$$

$$\mathcal{F}_k(a, b, \alpha) = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \, \mathrm{e}^{-(\lambda-1)/\tau} \, \lambda^{\alpha} \, (1-\zeta)^b {}_2\mathrm{F}_1\left[a, b; b+1; x_2\right]$$

$$I_{1,n} = \frac{2\gamma\gamma'}{\sqrt{\gamma^2 + \gamma'^2}} \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \, \mathrm{e}^{-(\lambda-1)/2} \, \zeta^{n+1} \, {}_2\mathrm{F}_1\left(\frac{1}{2}, n+1, x_1\right)$$

$$I_{2,n} = \frac{1}{\sqrt{2}} \sum_{k=0}^{n} \mathcal{D}(n, k, 1) \, \mathcal{F}_k\left(\frac{3}{2}, k + \frac{1}{2}, 0\right)$$

$$I_{3,n} = \frac{(\gamma^{-1} + \gamma'^{-1})}{\sqrt{2}} \sum_{k=0}^{n} \mathcal{D}(n, k, 1) \, \mathcal{F}_k\left(\frac{3}{2}, k + \frac{1}{2}, 1\right)$$

$$I_{4,n} = \frac{1}{\sqrt{2}(n+1)} \sum_{k=0}^{n+1} \mathcal{D}(n+1, k, 1) \, \mathcal{F}_k\left(\frac{3}{2}, k + \frac{1}{2}, 0\right)$$

$$I_{5,n} = \sqrt{2}\gamma\gamma' \sum_{k=0}^{n} \mathcal{D}(n, k, 2) \, \mathcal{F}_k\left(\frac{1}{2}, k + \frac{3}{2}, 0\right)$$

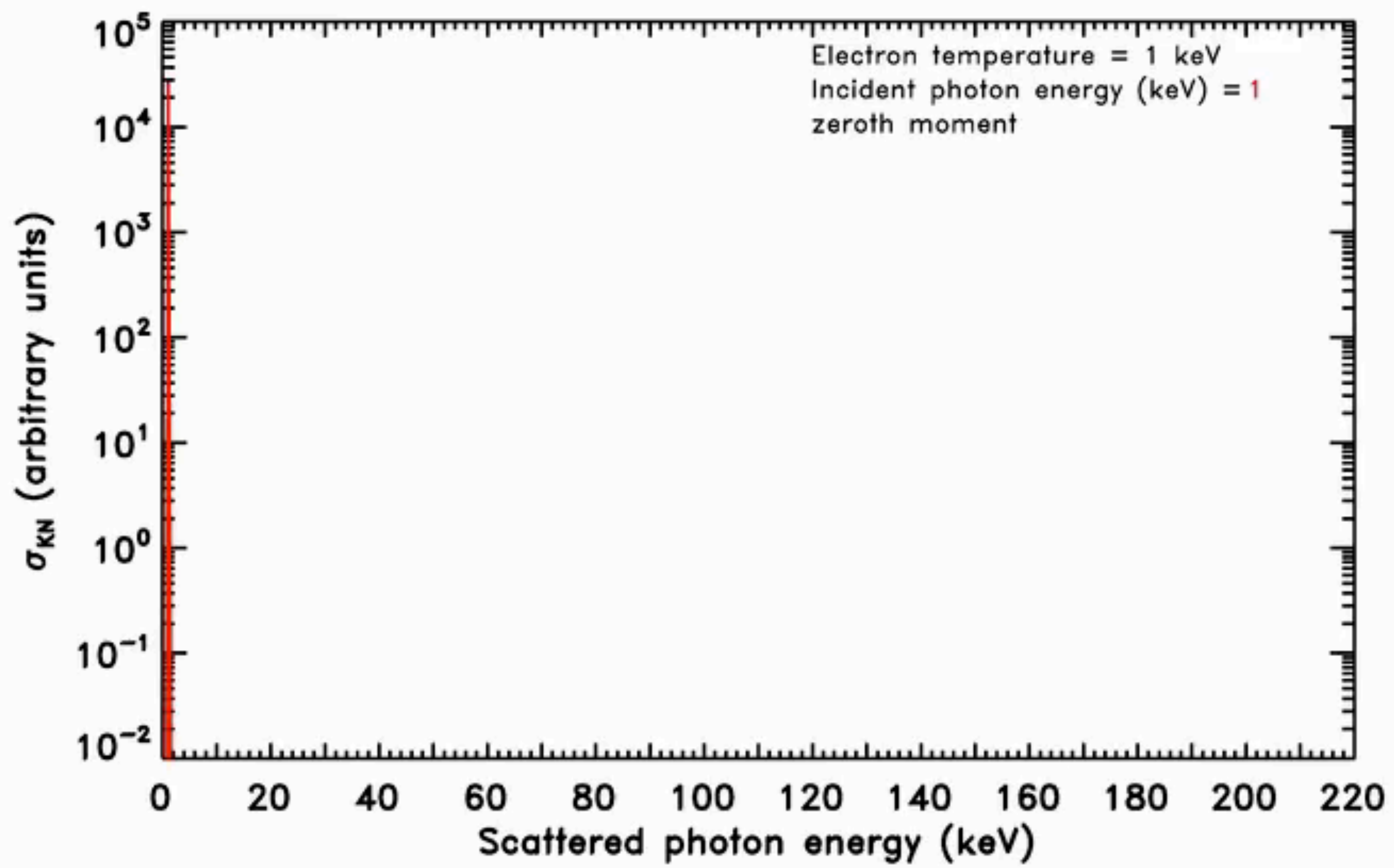
$$I_{6,n} = -2\sqrt{2} \sum_{k=0}^{n} \mathcal{D}(n, k, 1) \, \mathcal{F}_k\left(\frac{1}{2}, k + \frac{1}{2}, 0\right)$$

$$I_{7,n} = \frac{2\sqrt{2}}{\gamma\gamma'} \sum_{k=0}^{n} \mathcal{D}(n, k, 0) \, \mathcal{F}_k\left(\frac{1}{2}, k - \frac{1}{2}, 0\right)$$

$$\mathcal{D}(n, k, l) = \frac{(-1)^{k+1} {}_nC_k}{2k + 2l - 1}$$

The GRCS Kernel

(zeroth moment)



The GRCS Kernel

(zeroth moment)

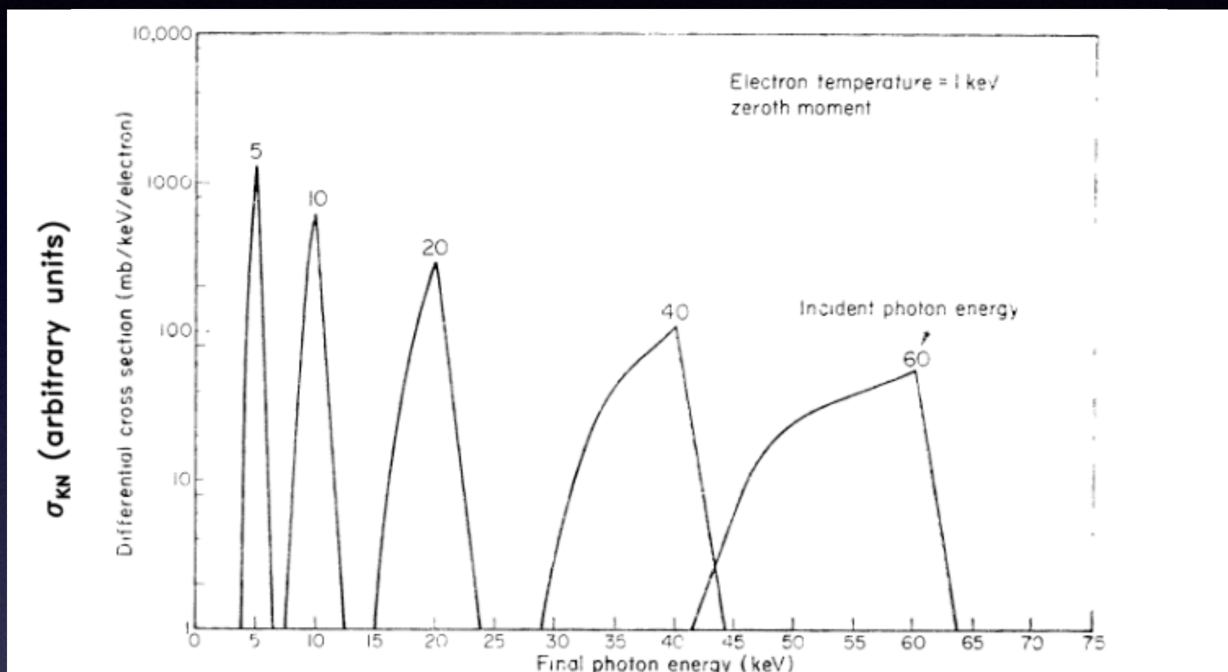


FIG. 8.1a. The differential scattering cross section: $n = 0$; $T = 1$.

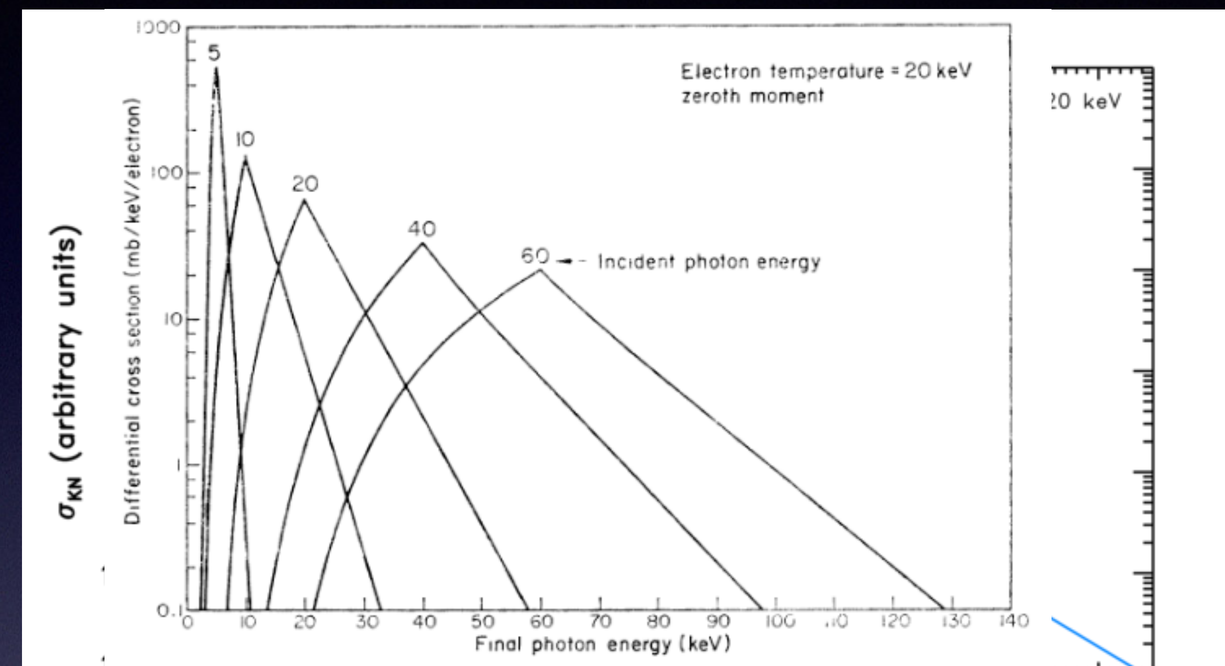
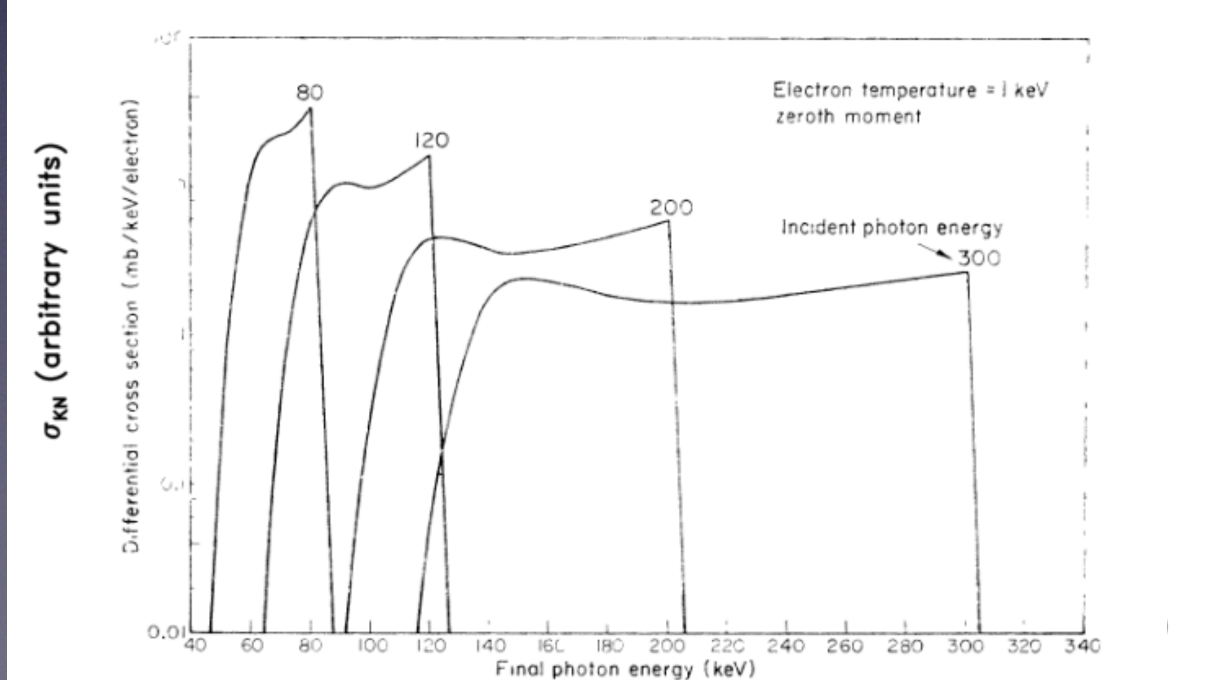
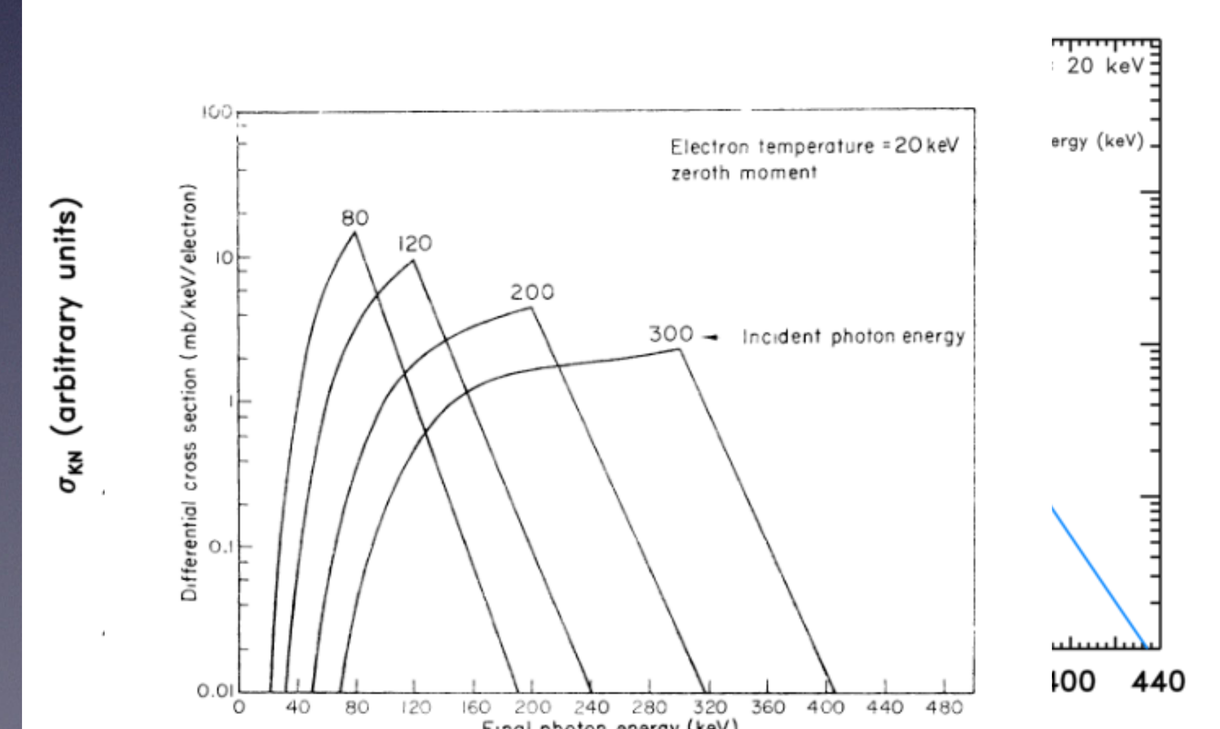


FIG. 8.2a. The differential scattering cross section: $n = 0$; $T = 20$.

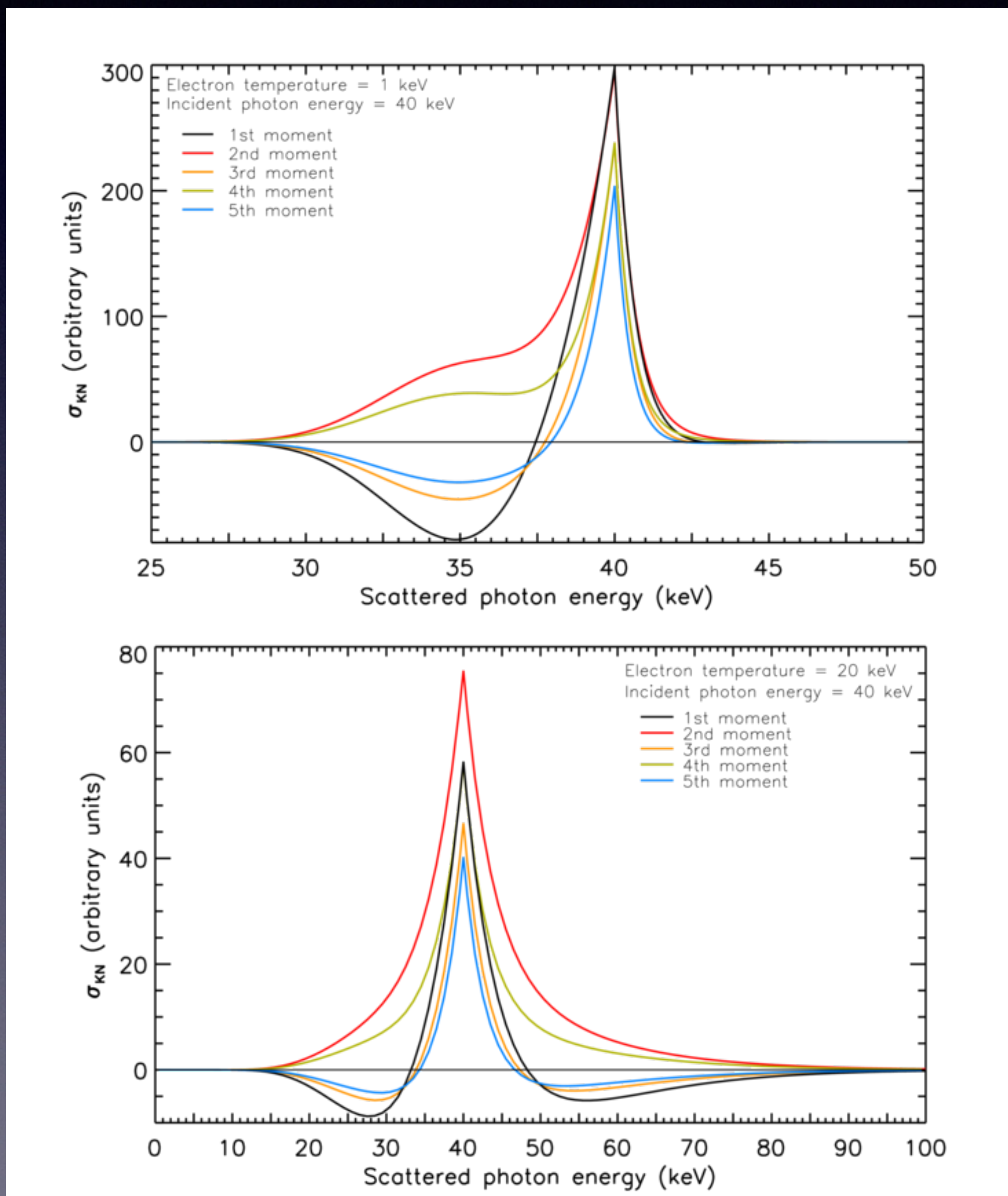


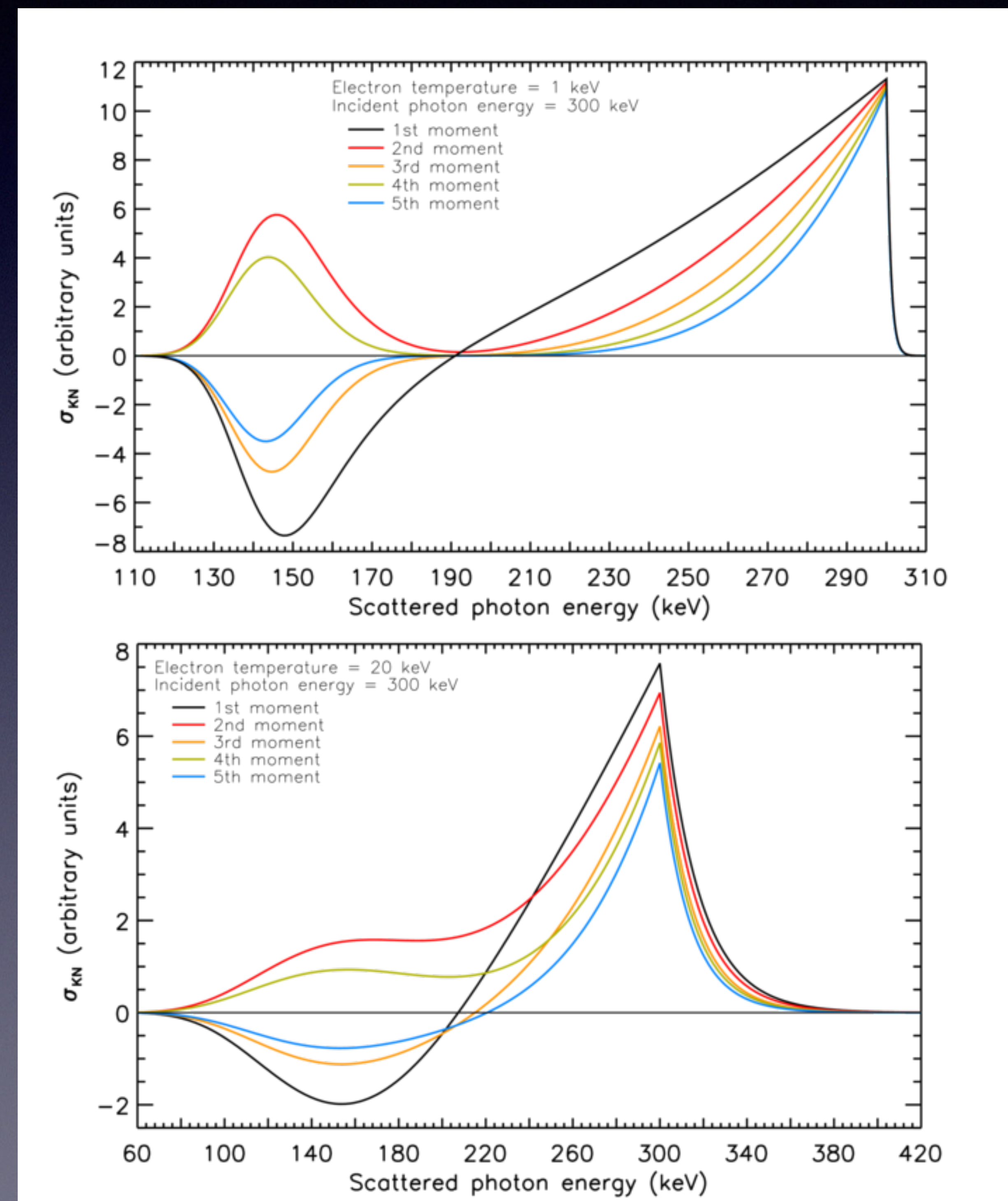


Pomraning 1972

The GRCS Kernel

(1st - 5th moments)





Conclusions

- GRRT is a powerful tool to calculate the observed images and EM emission in general relativistic environments
- The structure of the accretion flow significantly alters both the images and the spectrum
- Radiative transfer calculations can deal with the combined relativistic, geometrical, optical and physical effects
- Hard to determine key black hole parameters from emission spectrum - strongly dependent on many physical effects
- Future work must focus on more comprehensive treatment of both radiation processes and the accretion flow

Future Work

- Re-formulate geodesic equations in Kerr-Schild form, removing stiffness at event horizon
- Construct interface between GRRT and GRMHD simulations
- Parallelize code in MPI (trivial in OpenMP)
- Consider more radiation processes
- Proper treatment of scattering
- Polarization