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COCAL and binary neutron star initial data

Antonios Tsokaros
Goethe University
Institute of Theoretical Physics

in collaboration with
Kōji Uryū, University of Ryukyus, Department of Physics, Japan, and
Luciano Rezzolla, Goethe University, Institute of Theoretical Physics, Germany.

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Initial data means initial value problem

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ $\qquad R_{\alpha\beta} = \partial_\mu \Gamma^\mu{}_{\alpha\beta} - \partial_\alpha \Gamma^\mu{}_{\mu\beta} + \Gamma^\mu{}_{\alpha\beta}\Gamma^\nu{}_{\nu\mu} - \Gamma^\mu{}_{\alpha\nu}\Gamma^\nu{}_{\mu\beta}$

$$\begin{aligned} G_{\mu\nu} = & \frac{1}{2} g^{\alpha\beta}(\partial_\beta\partial_\nu g_{\mu\alpha} + \partial_\beta\partial_\mu g_{\nu\alpha} - \partial_\alpha\partial_\beta g_{\mu\nu} - \partial_\mu\partial_\nu g_{\alpha\beta}) \\ & + \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} g^{\rho\sigma}(\partial_\alpha\partial_\beta g_{\rho\sigma} - \partial_\beta\partial_\rho g_{\sigma\alpha}) + f_{\mu\nu}(g, \partial g) \end{aligned}$$

Theorem: `Cauchy-Kowalewski`

Given the system

$$\frac{\partial^2 g_{\mu\nu}}{\partial t^2} = F_{\mu\nu}\left(t, x, y, z; g_{\alpha\beta}, \frac{\partial g_{\alpha\beta}}{\partial t}, \frac{\partial g_{\alpha\beta}}{\partial x^b}, \frac{\partial^2 g_{\alpha\beta}}{\partial t \partial x^b}, \frac{\partial^2 g_{\alpha\beta}}{\partial x^b \partial x^c}\right)$$

where $F_{\mu\nu}$ analytic function of its variables, and given the initial data

$$g_{\mu\nu}(t_0, x, y, z) = A_{\mu\nu}(x, y, z), \qquad \frac{\partial g_{\mu\nu}}{\partial t}(t_0, x, y, z) = B_{\mu\nu}(x, y, z)$$

then there exists unique analytic solution (in a ngh of t_0).

Analysis of the equations

Problem: We DO NOT have 10 equations with $\frac{\partial^2 g_{\alpha\beta}}{\partial t^2}$, but ONLY 6.
Components G_{00}, G_{01}, G_{02}, G_{03} do not contain ANY second time derivatives, they depend only on the initial data.

Structure of Einstein equations

$$\boxed{\text{4 constraints eqs}} \qquad + \qquad \boxed{\text{6 second order evolution eqs}}$$

Einstein system is underdetermined. This underdetermination is not physical.

Similar in Maxwell system $\partial^\alpha(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = 0$. We have 3 evolution and 1 constraint (Gauss law) eqs. But $\{A_\mu, \frac{\partial A_\mu}{\partial t}\}$ determine a unique solution up to a gauge. Use Lorentz gauge to bring Maxwell system in C-K form.

<u>A geometric evolution problem:</u> Can the Einstein system be written as an initial value problem so as we can apply C-K? and if yes, what are going to be the dynamical functions?

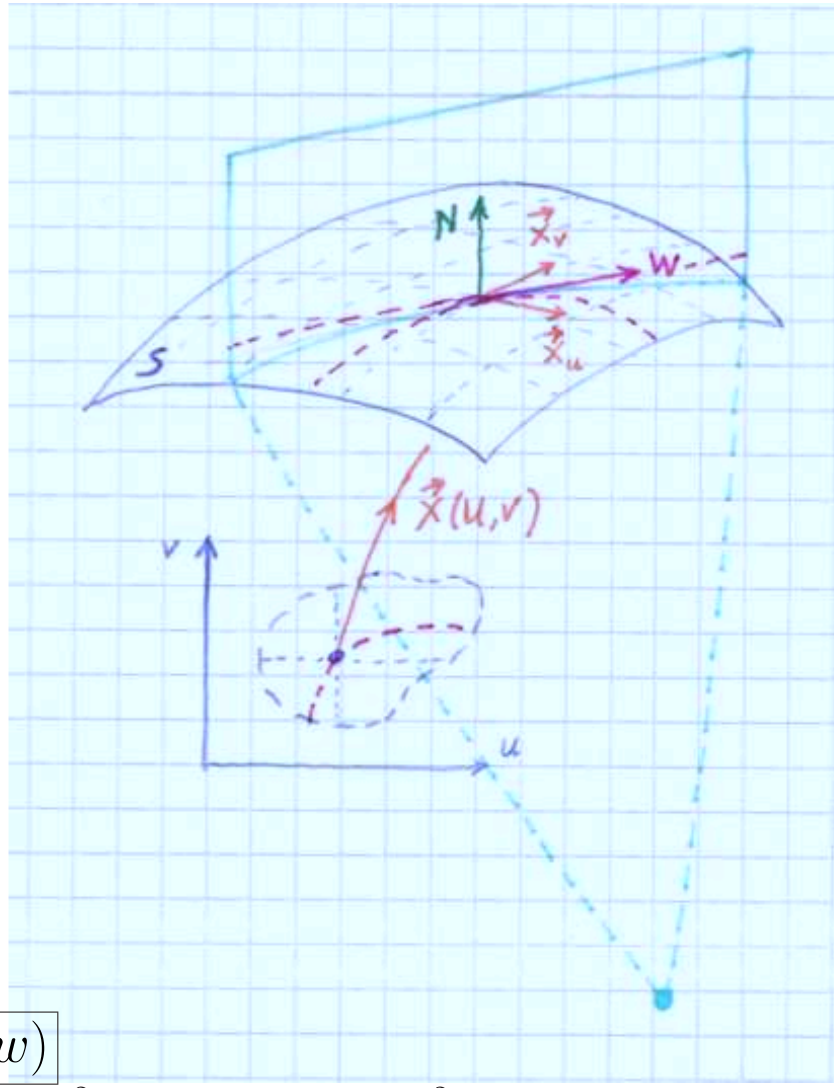
<u>A spacetime identification problem:</u> How can we identify a spacetime, as a solution of the Einstein eqs, in other words how can we tell that two phenomenally different spacetimes are essentially the same?

Two dimensional Riemannian analogue

Regular surface S: $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$

$E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v$

Any tangent vector: $w = u_0 \mathbf{x}_u + v_0 \mathbf{x}_v$



First Fundamental Form(FFF): $I(w) = w \cdot w$

measures lengths $I(w) = E u_0^2 + 2F u_0 v_0 + G v_0^2$

Metric or FFF: $\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$

Gauss map: $N : S \to S^2$

$$N \circ \mathbf{x} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$$

Weingarten map: $\mathbf{S} : T_p(S) \to T_p(S)$

$$\mathbf{S}(w) = -dN(w)$$

Second Fundamental Form(SFF): $II(w) = -w \cdot dN(w)$

measures the curvature of the normal section $II(w) = e u_0^2 + 2f u_0 v_0 + g v_0^2$

$e = -N_u \cdot \mathbf{x}_u = N \cdot \mathbf{x}_{uu}$

$f = -N_v \cdot \mathbf{x}_u = N \cdot \mathbf{x}_{uv}$

$g = -N_v \cdot \mathbf{x}_v = N \cdot \mathbf{x}_{vv}$

SFF: $\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} e & f \\ f & g \end{bmatrix}$

Gauss-Weigarten eqs:

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma^1{}_{11}\mathbf{x}_u + \Gamma^2{}_{11}\mathbf{x}_v + eN\\
\mathbf{x}_{uv} &= \Gamma^1{}_{12}\mathbf{x}_u + \Gamma^2{}_{12}\mathbf{x}_v + fN\\
\mathbf{x}_{vu} &= \Gamma^1{}_{21}\mathbf{x}_u + \Gamma^2{}_{21}\mathbf{x}_v + fN\\
\mathbf{x}_{vv} &= \Gamma^1{}_{22}\mathbf{x}_u + \Gamma^2{}_{22}\mathbf{x}_v + gN\\
N_u &= -K_1^1\mathbf{x}_u - K_1^2\mathbf{x}_v\\
N_v &= -K_2^1\mathbf{x}_u - K_2^2\mathbf{x}_v
\end{aligned}$$

$\Gamma^1{}_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG-F^2)}, \ldots$

$K_1^1 = -\frac{fF-eG}{EG-F^2}, \ldots$

PDE of surfaces $\{\mathbf{x}_u, \mathbf{x}_v, N\}$

$$\begin{aligned}
\mathbf{x}_{ij} &= \Gamma^k{}_{ij}\mathbf{x}_k + K_{ij}N \quad (Gauss\ eqs)\\
N_i &= -K_i^j\mathbf{x}_j \qquad (Weingarten\ eqs)
\end{aligned}$$

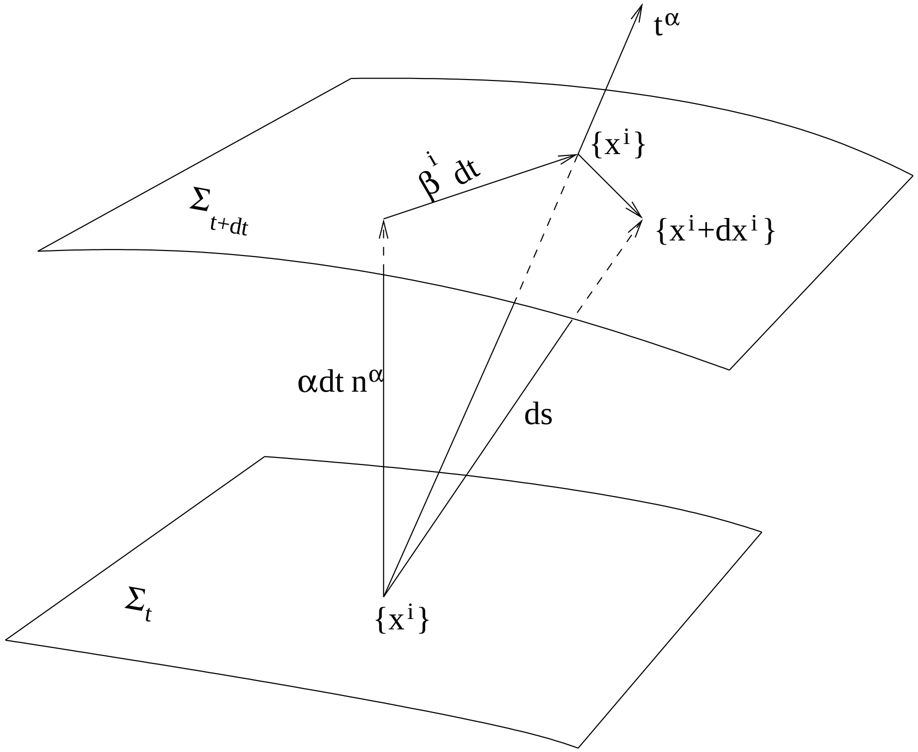
15 eqs for 9 functions. But...

Compatibility eqs: $\mathbf{x}_{ijk} = \mathbf{x}_{ikj}$ & $N_{uv} = N_{vu}$

$$\begin{aligned}
-E\frac{eg-f^2}{EG-F^2} &= (\Gamma^2{}_{12})_u - (\Gamma^2{}_{11})_v + \Gamma^1{}_{12}\Gamma^2{}_{11} + \Gamma^2{}_{12}\Gamma^2{}_{12} - \Gamma^2{}_{11}\Gamma^2{}_{22} - \Gamma^1{}_{11}\Gamma^2{}_{12}\\
e_v - f_u &= e\Gamma^1{}_{12} + f(\Gamma^2{}_{12} - \Gamma^1{}_{11}) - g\Gamma^2{}_{11}\\
f_v - g_u &= e\Gamma^1{}_{22} + f(\Gamma^2{}_{22} - \Gamma^1{}_{12}) - g\Gamma^2{}_{12}
\end{aligned}$$

(`Bonnet, Fundamental theorem of surfaces:`) Given E, F, G, e, f, g and assuming that they satisfy the Gauss and Mainardi-Codazzi eqs, there exists $\mathbf{x}$ such that the regular surface has E, F, G, e, f, g as coefficients of the FFF and SFF. Also any other solution $\mathbf{y}$ that satisfies the same conditions differs from $\mathbf{x}$ by an isometry.

The initial value problem in general relativity



$$D_\mathbf{u}\mathbf{w} = \nabla_\mathbf{u}\mathbf{w} + \mathbf{K}(\mathbf{u},\mathbf{w})\mathbf{n}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$\alpha = \alpha(t,x,y,z)$ the lapse, shows how time coordinate evolves

$\beta^i = \beta^i(t,x,y,z)$ shift vector, shows how spatial coord. evolve

$\boxed{\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta}$ First Fundamental Form of 3-geometry

$\boxed{K_{\alpha\beta} = -D_\alpha n_\beta = -\gamma_\alpha{}^\mu \gamma_\beta{}^\nu \nabla_\mu n_\nu}$ Second Fundamental Form of 3 geometry

Initial data 12 functions: $\{\gamma_{ij}, K_{ij}\}$.

Constraint equations

$$\mathcal{R} - K_{ij}K^{ij} + K^2 = 16\pi\rho_H$$

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i$$

where

$$\rho_H = T^{\alpha\beta} n_\alpha n_\beta,$$

$$S^i = -\gamma^{ij} n^\alpha T_{\alpha j}$$

Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + 2D_i\beta_j + 2D_j\beta_i$$

$$\partial_t K_{ij} = -D_iD_j\alpha + \alpha(\mathcal{R}_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) + $$
$$+\beta^k\partial_k K_{ij} + K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k -$$
$$-8\pi\alpha(S_{ij} - \tfrac{1}{2}\gamma_{ij}(S - \rho_H))$$

where $S_{ij} = \gamma_{i\alpha}\gamma_{j\beta}T^{\alpha\beta}$ and $S = \gamma^{ij}S_{ij}$

Solving the constraint equations: Conformal methods

Under a conformal transformation $\boxed{\gamma_{ij} = \psi^\lambda \tilde{\gamma}_{ij}}$ we have

- The Hamiltonian constraint

$$\boxed{\tilde{D}_i \tilde{D}^i \psi = \frac{\psi}{2\lambda}\tilde{\mathcal{R}} - \frac{\lambda - 4}{4\psi}\tilde{D}^i\psi \tilde{D}_i\psi - \frac{\psi^{\lambda+1}}{2\lambda}K_{ij}K^{ij} + \frac{\psi^{\lambda+1}}{2\lambda}K^2 - \frac{8\pi\rho_H\psi^{\lambda+1}}{\lambda}}$$

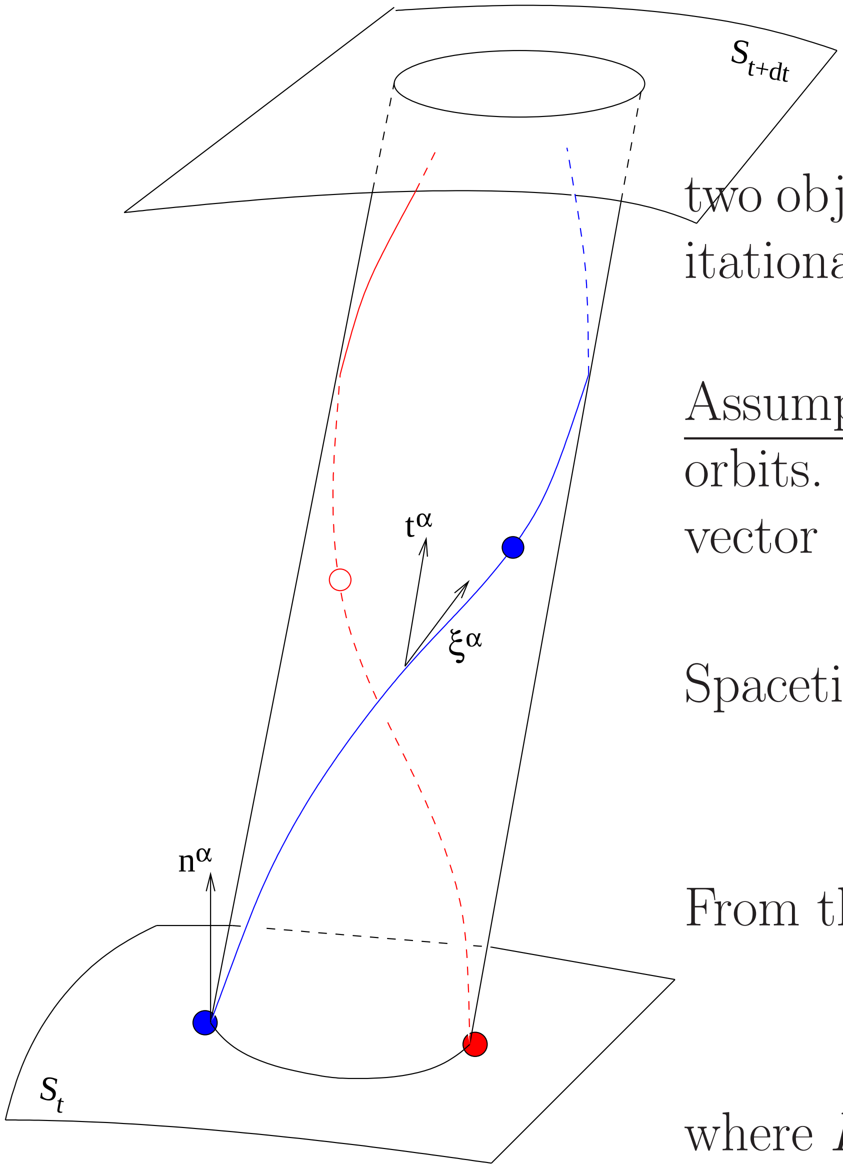
- The momentum constraint

$$\tilde{D}_j K^{ij} + \frac{5\lambda}{2\psi}K^{ij}\tilde{D}_j\psi - \frac{\lambda}{2\psi^{\lambda+1}}K\tilde{D}^i\psi - \psi^{-\lambda}\tilde{D}^i K = 8\pi j^i$$

- The spatial trace of the second evolution equation is written

$$\boxed{\tilde{D}_i\tilde{D}^i\alpha = -\frac{\lambda}{2\psi}\tilde{D}_i\psi\tilde{D}^i\alpha + \psi^\lambda\alpha[K_{ij}K^{ij} + 4\pi(\rho_H + S)) - \mathcal{L}_{\mathbf{n}}(K)]}$$

Solving the constraint equations for a binary system



Real physics: because of radiation reaction the two objects are spiraling towards each other. Also gravitational radiation reaction circularizes the orbits.

Assumption: neglect the in-spiraling and assume closed orbits. In other words assume the existence of a helical vector

$$\xi^\mu = (\partial_t)^\mu + \Omega(\partial_\phi)^\mu$$

Spacetime symmetry:

$$\mathcal{L}_{\boldsymbol{\xi}}\gamma_{ij} = 0, \qquad \mathcal{L}_{\boldsymbol{\xi}}K_{ij} = 0.$$

From the first evolution equation ⇒

$$K_{ij} = \frac{1}{2\alpha}(D_iB_j + D_jB_i)$$

where $B^i = \beta^i + \Omega\phi^i$ the corotating shift.

Fluid equations - Stationary equilibria

$$T_{\alpha\beta} = (\rho + P)u_\alpha u_\beta + P g_{\alpha\beta} = \rho_0 h u_\alpha u_\beta + P g_{\alpha\beta}, \qquad h = \frac{\rho+P}{\rho_0} \text{ relativistic enthalpy}$$

$$\boxed{u^\beta \nabla_\beta (h u_\alpha) + \nabla_\alpha h = 0} \quad \text{and} \quad \boxed{\nabla_\alpha(\rho_0 u^\alpha) = 0}$$

The 4-velocity is decomposed as

$$u^\alpha = u^t(\xi^\alpha + V^\alpha), \qquad \xi^\alpha = \alpha n^\alpha + B^\alpha, \qquad \hat{u}_i = \gamma_i^\alpha h u_\alpha$$

<u>Go to 3+1 and exhibit the Lie derivative wrt ξ</u>

$$\gamma_i^\alpha \mathcal{L}_{\boldsymbol{\xi}}(h u_\alpha) + D_i\left(\frac{h}{u^t} + \hat{u}_j V^j\right) + V^j(D_j(\hat{u}_i) - D_i(\hat{u}_j)) = 0 \qquad \text{Euler on the slice}$$

and

$$\mathcal{L}_{\boldsymbol{\xi}}(\rho_0 u^t) + \frac{1}{\alpha} D_i(\alpha \rho_0 u^t V^i) = 0 \qquad \text{Rest mass conservation on the slice}$$

- <u>case 1</u>: Corotating binaries. $V^i = 0$ thus $u^\alpha = u^t \xi^\alpha$
- <u>case 2</u>: Irrotational binaries. $\hat{u}_i = D_i \Phi \quad \Rightarrow \quad V^i = \frac{1}{h u^t} D^i \Phi - B^i$
- <u>case 3</u>: Spinning binaries. $\hat{u}_i = D_i \Phi + w_i \quad \Rightarrow \quad V^i = \frac{1}{h u^t}(D^i \Phi + w^i) - B^i$

recap: Einstein-Euler system: 6 potentials + 2 constants

$$\nabla^2\psi = \left[-\frac{\psi^5}{32\alpha^2}(L\beta)_{ij}(L\beta)^{ij}\right] + 2\pi\rho_H\psi^5 \qquad = S_{g\psi} + S_{f\psi} = S_\psi$$

$$\nabla^2\alpha = \left[-\frac{2}{\psi}\partial_i\psi\partial^i\alpha + \frac{\psi^4}{4\alpha}(L\beta)_{ij}(L\beta)^{ij}\right] + 4\pi\alpha\psi^4(\rho_H + S) \qquad = S_{g\alpha} + S_{f\alpha} = S_\alpha$$

$$\nabla^2\beta^i = \left[-\frac{1}{3}\partial^i\partial_j\beta^j + \partial_j \ln\left(\frac{\alpha}{\psi^6}\right)(L\beta)^{ij}\right] + 16\pi\alpha\psi^4 j^i \qquad = S_{g\beta} + S_{f\beta} = S_\beta$$

$$\nabla^2\Phi = \psi^4\left\{\partial_i(hu^tB^i - w^i) + (hu^tB^i - w^i - \frac{\psi^{-4}}{3}\partial^i\Phi)\partial_i \ln\psi^6 + \right.$$

$$\left. + \ (hu^tB^i - w^i - \psi^{-4}\partial^i\Phi)\partial_i \ln\left(\frac{\alpha\rho_0}{h}\right)\right\} \qquad = S_\Phi$$

with $(L\beta)_{ij} = \partial_i\beta_j + \partial_j\beta_i - \frac{2}{3}f_{ij}\partial_m\beta^m$.

Sources:

$$\rho_H = \frac{\rho_0\alpha^2}{h}(hu^t)^2 - P, \qquad S = \frac{\rho_0\alpha^2}{h}(hu^t)^2 - \rho_0 h + 3P, \qquad j^i = \rho_0\alpha u^t(\psi^{-4}\partial^i\Phi + w^i)$$

$$B^i = \beta^i + \Omega\phi^i \qquad \lambda = C + B^i\partial_i\Phi$$

Inverting the Laplacian: KEH iteration scheme

The field eqs are written in the form

$$\nabla^2\Phi = S(\Phi)$$

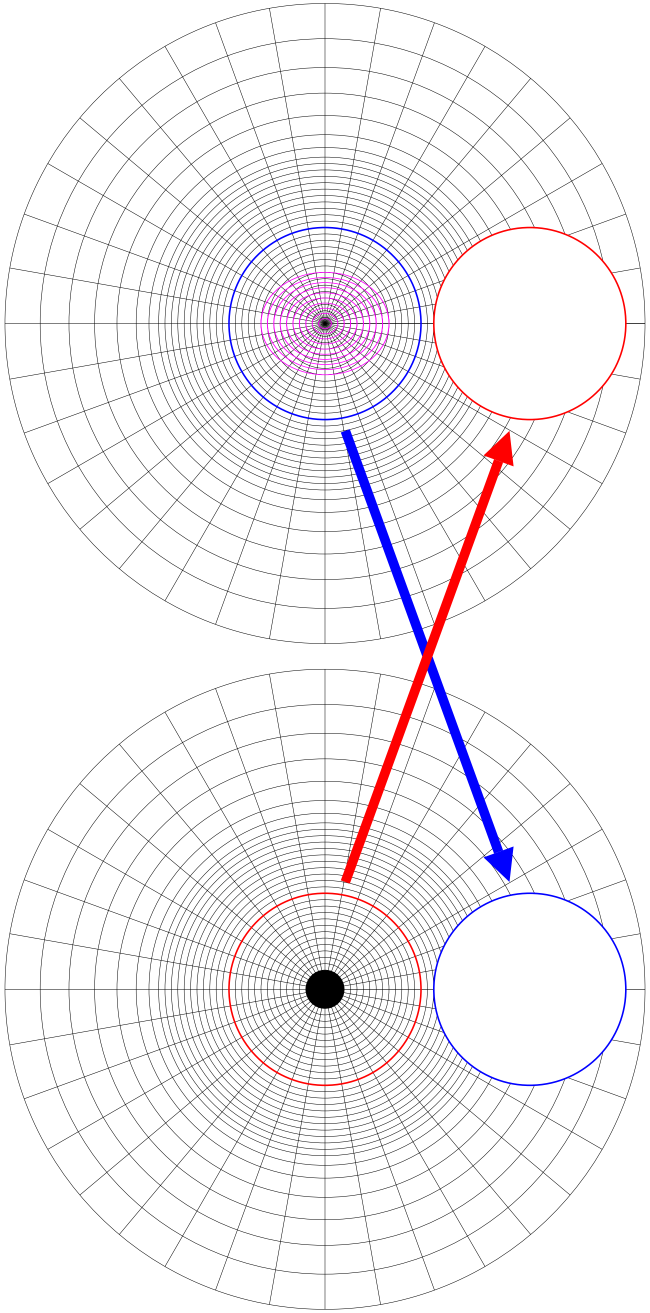
and solved iteratively by using the representation formula of pde.

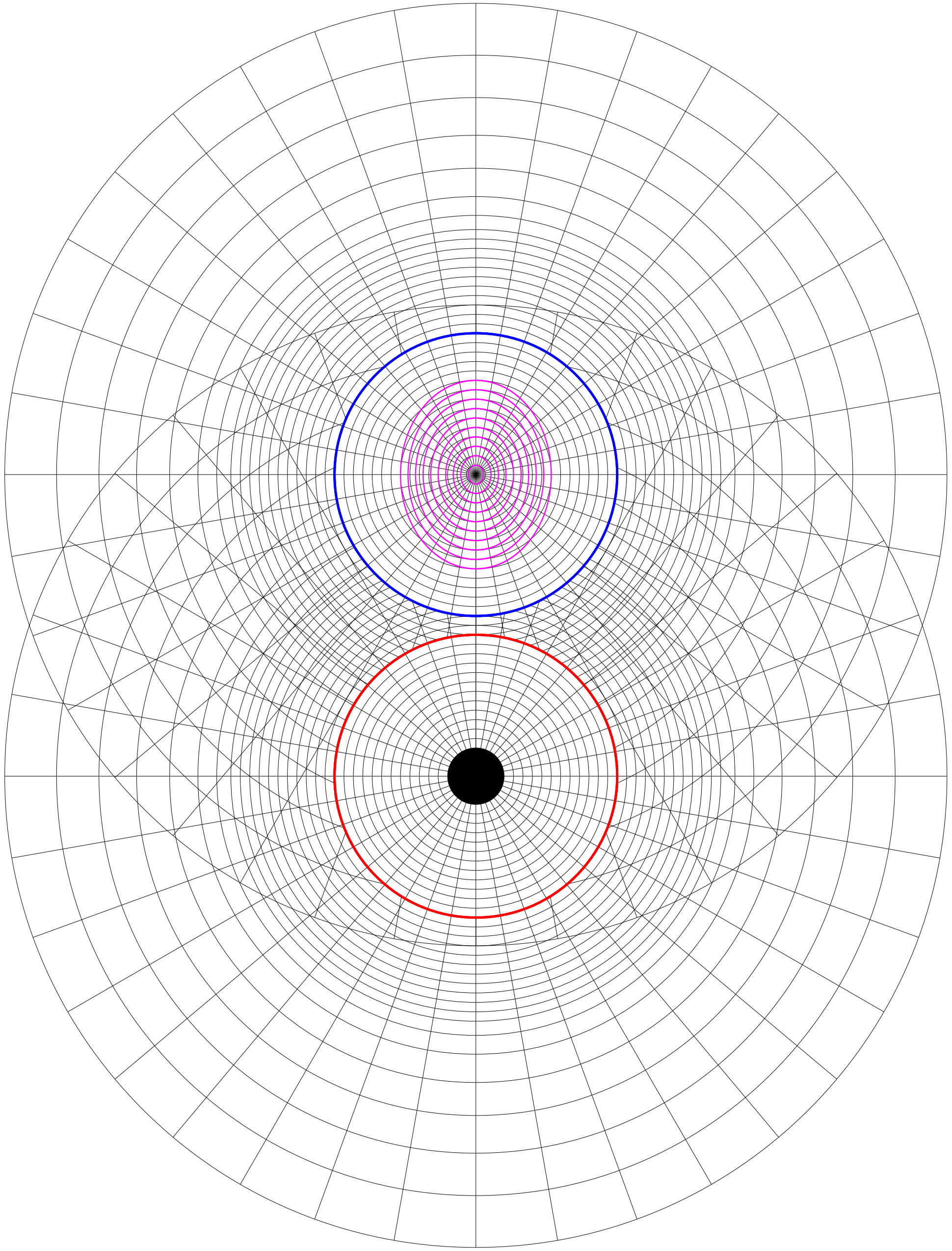
$$\Phi(x) = -\frac{1}{4\pi}\int_V G(x,x')S(x')d^3x' + \frac{1}{4\pi}\int_{\partial V}\left[G(x,x')\frac{\partial\Phi(x')}{\partial n'} - \Phi(x')\frac{\partial}{\partial n'}G(x,x')\right]dS'$$

Issues: V, ∂V, $G(x,x')$.

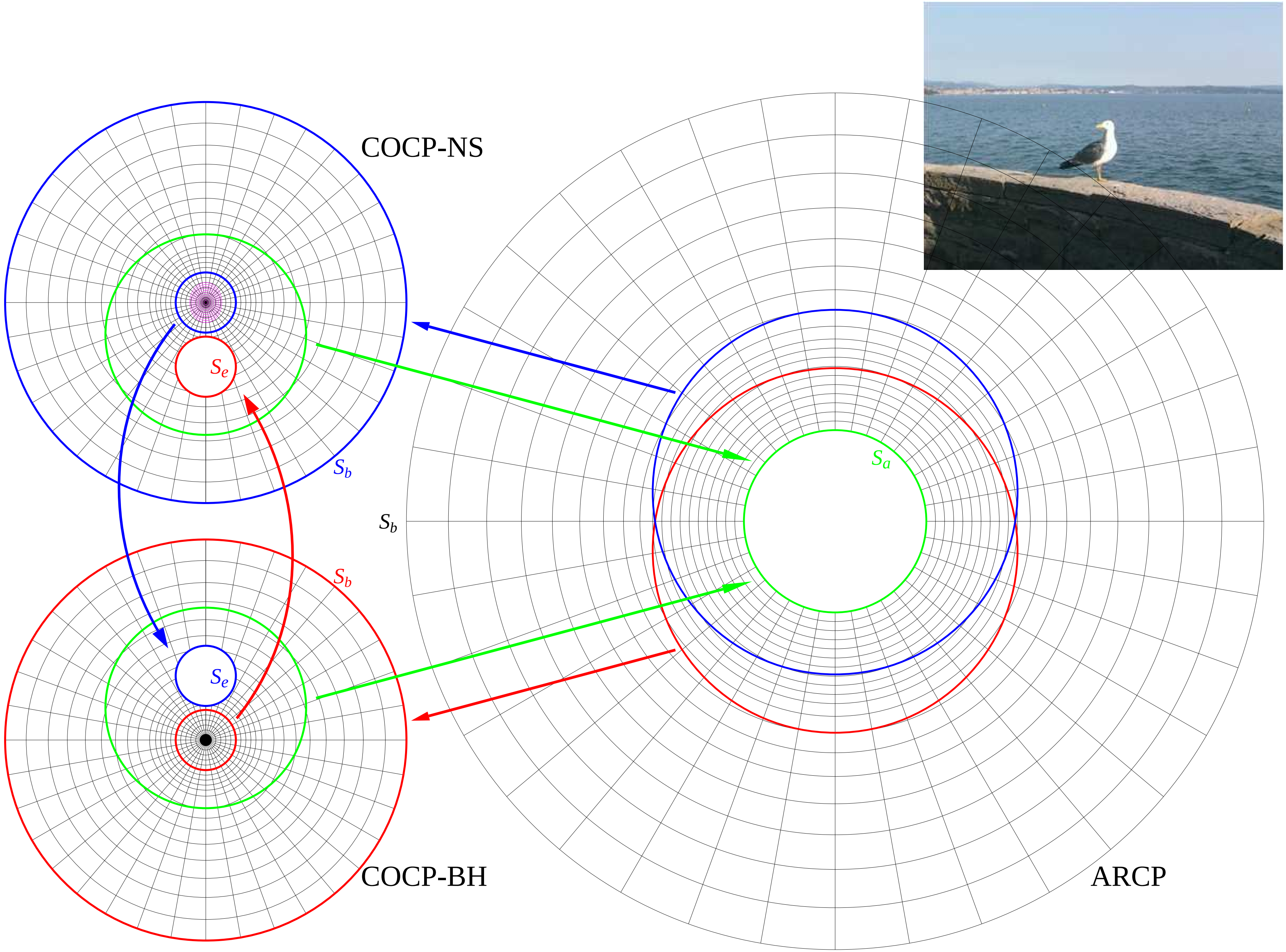
• V : At least 3 coordinate systems. One around each compact object, and one to cover the asymptotic region.
•∂V : Multiple boundaries (2-3 for each coordinate system). Special care for black holes.
•G : Depends on the shape of V and the boundaries ∂V. Different choices have been implemented. Important fact: The convergence of the iteration scheme depends crucially on the choice of the Green's function in conjunction with the boundary conditions.

Setup for the binary grids

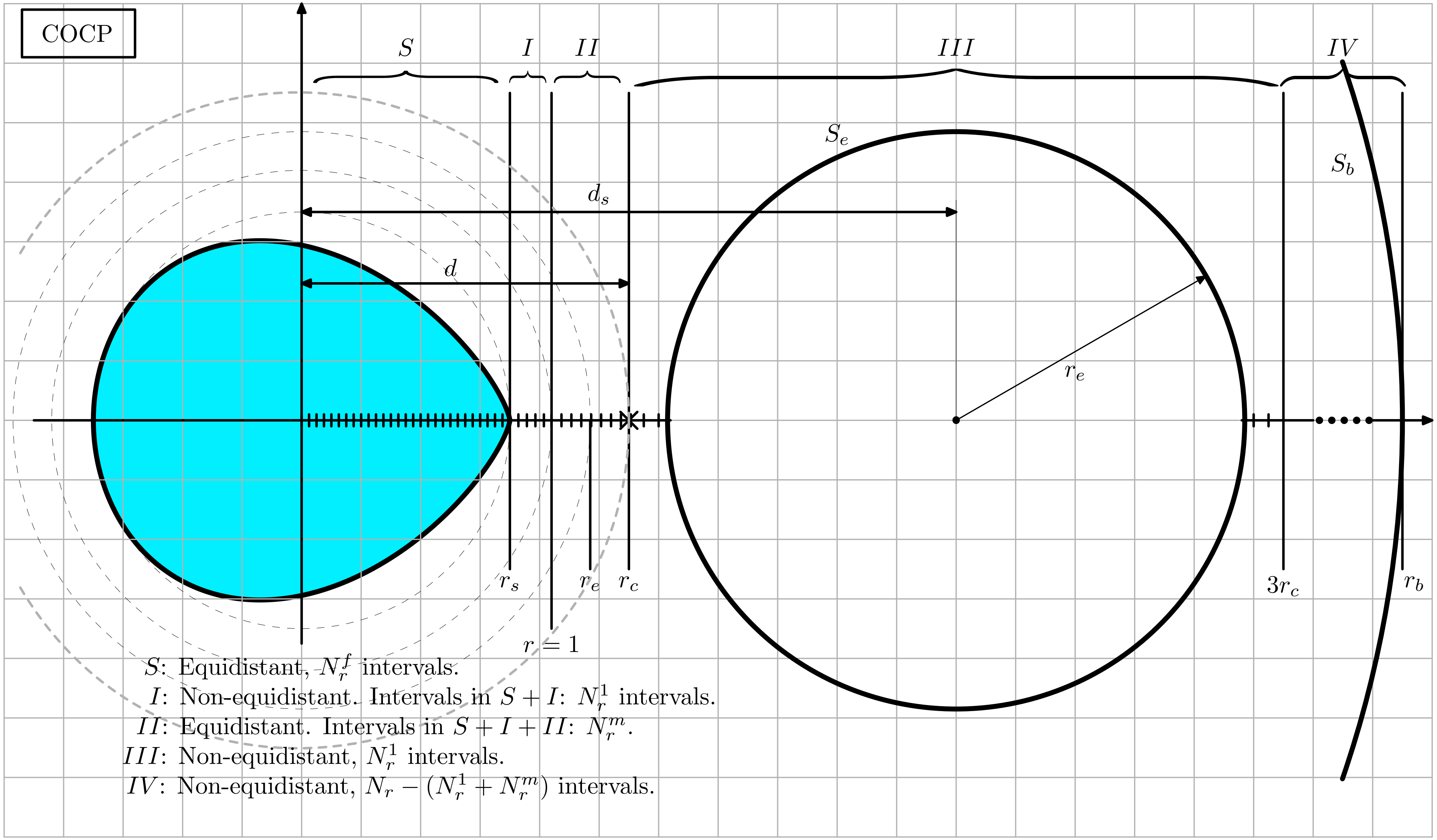




Compact Object CALculator = COCAL



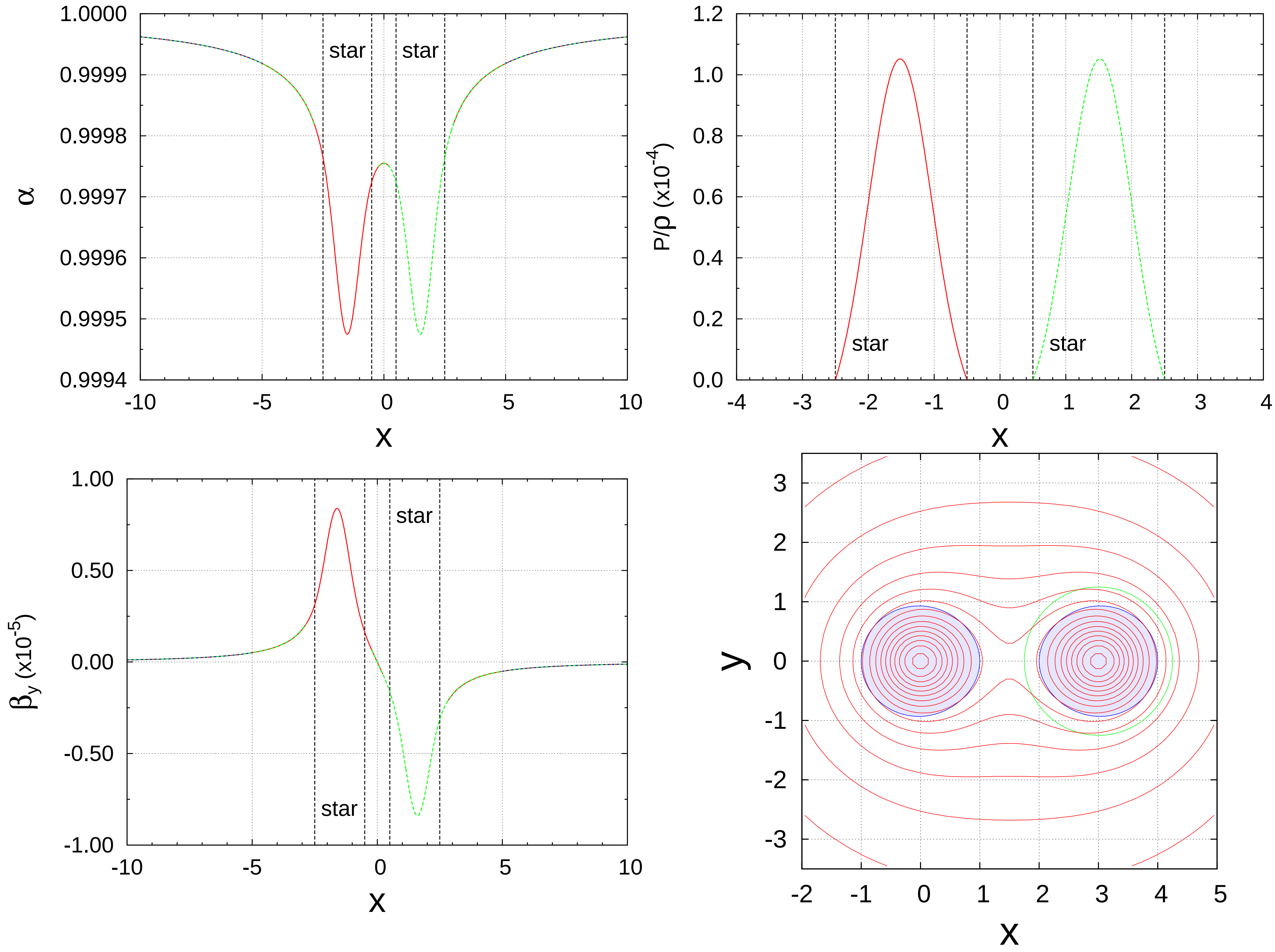
Gravitational grid in COCP and surface fitted coordinates



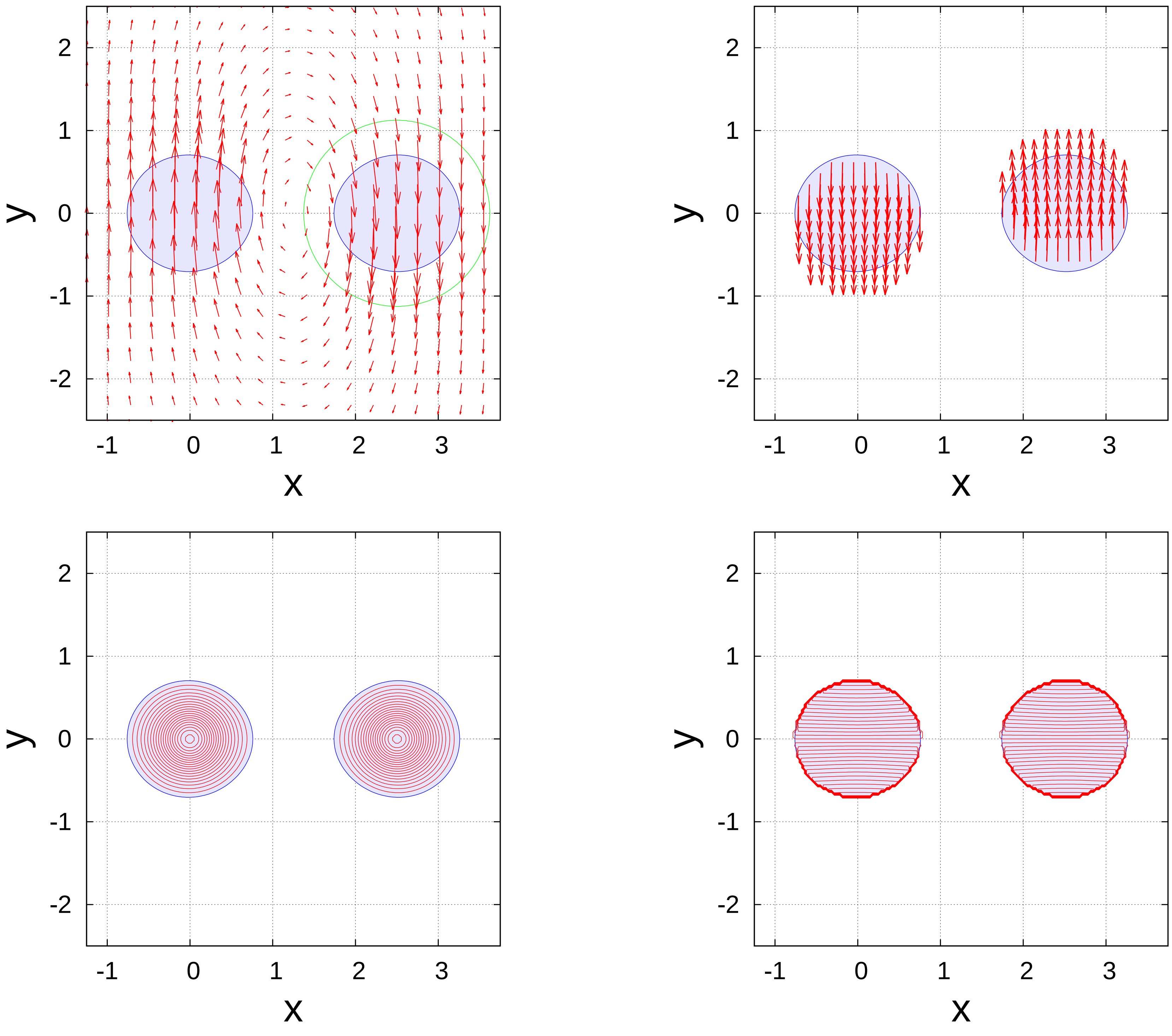
Fluid coordinates $\quad r_f = \dfrac{r}{R(\theta,\phi)}, \quad \theta_f = \theta, \quad \phi_f = \phi$

Rescaling factor $\quad R_0 : \quad \dfrac{R(\pi/2,0)}{R_0} = r_s = \dfrac{R(\pi/2,\pi)}{R_0}$

A binary corotating white dwarf solution: C=0.0002

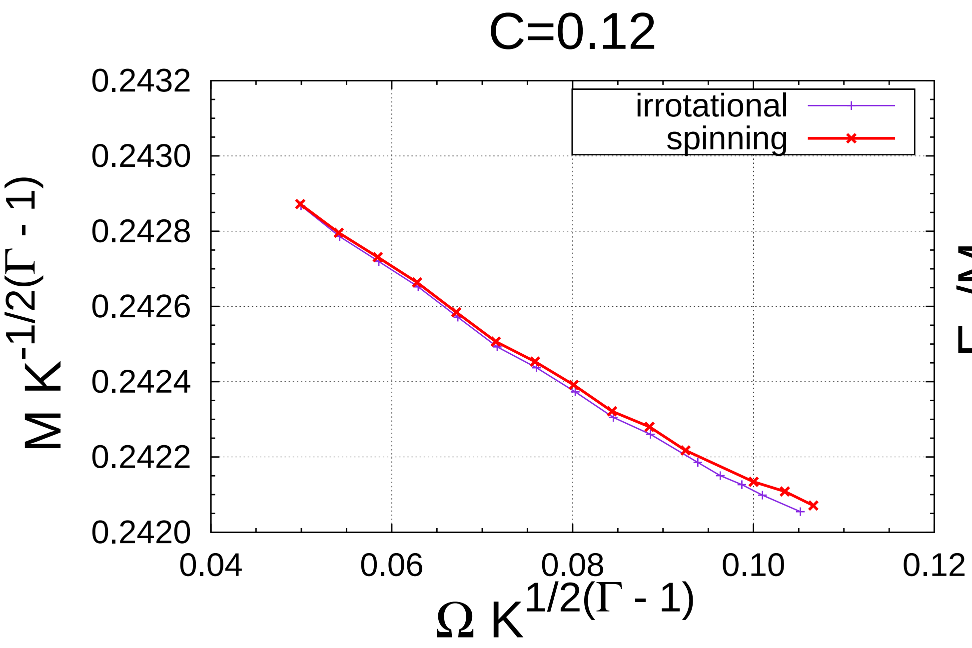


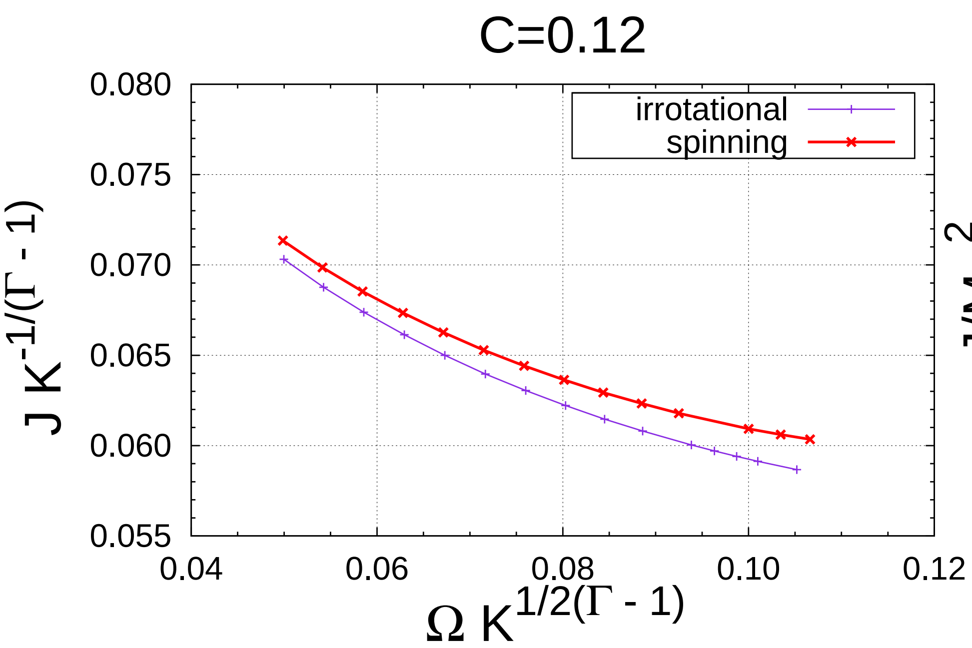
A binary irrotating neutron star solution: C=0.18



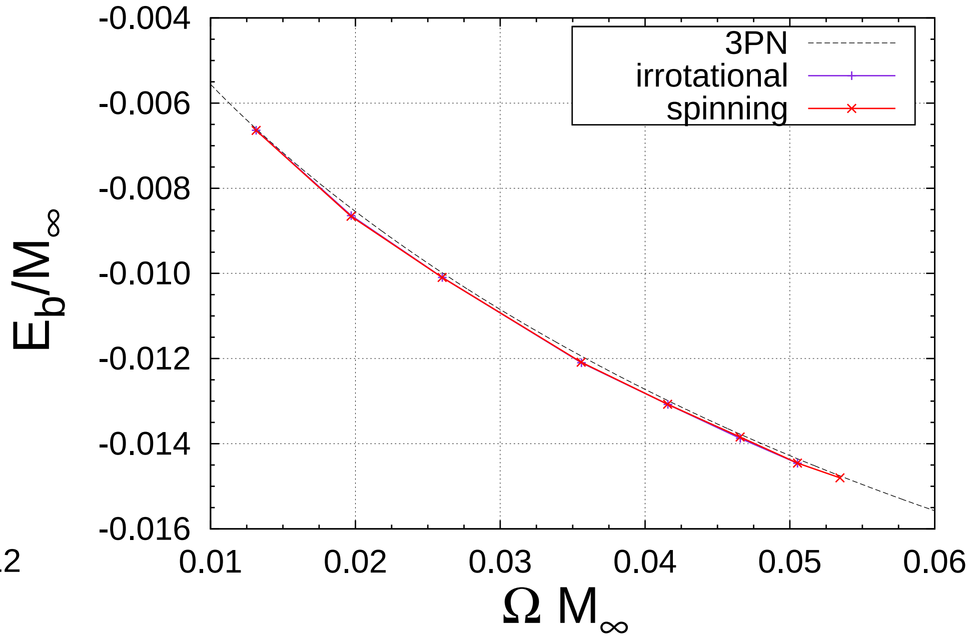
Quasi-equilibrium spinning sequences

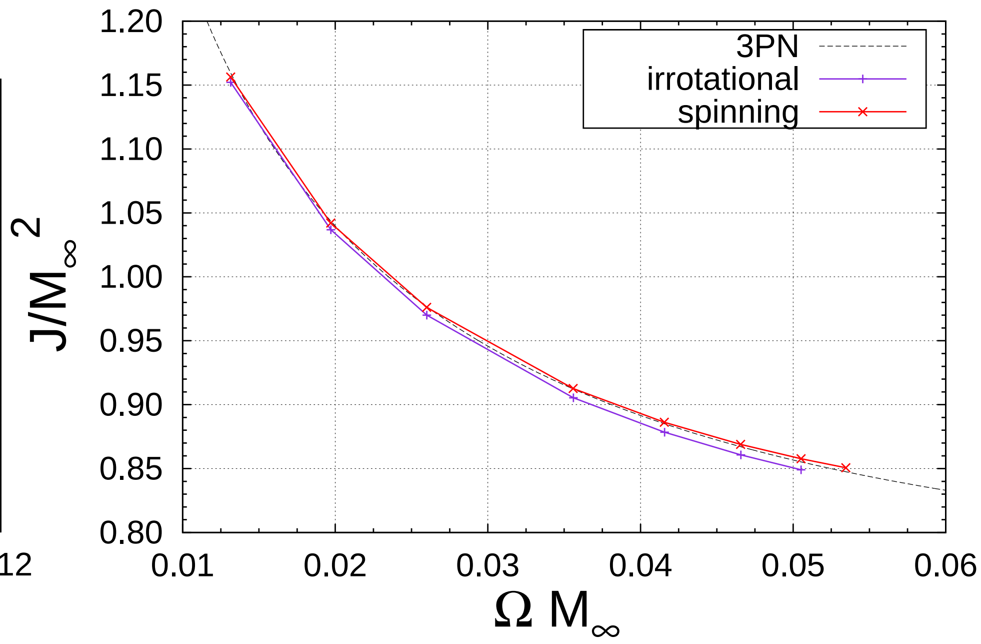
Simple polytrope:



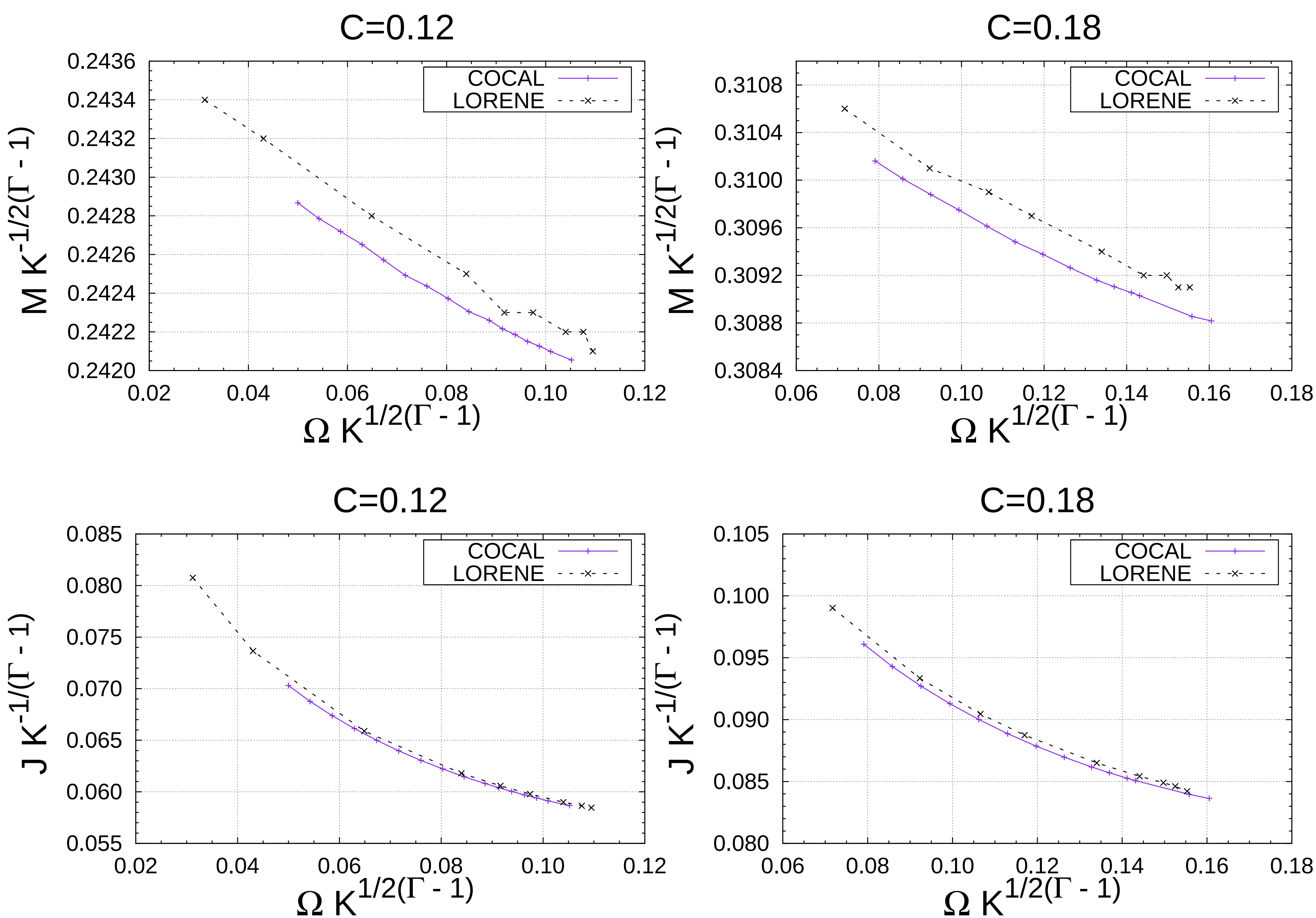


Piecewise polytrope, APR1: $M = 1.35 M_\odot$

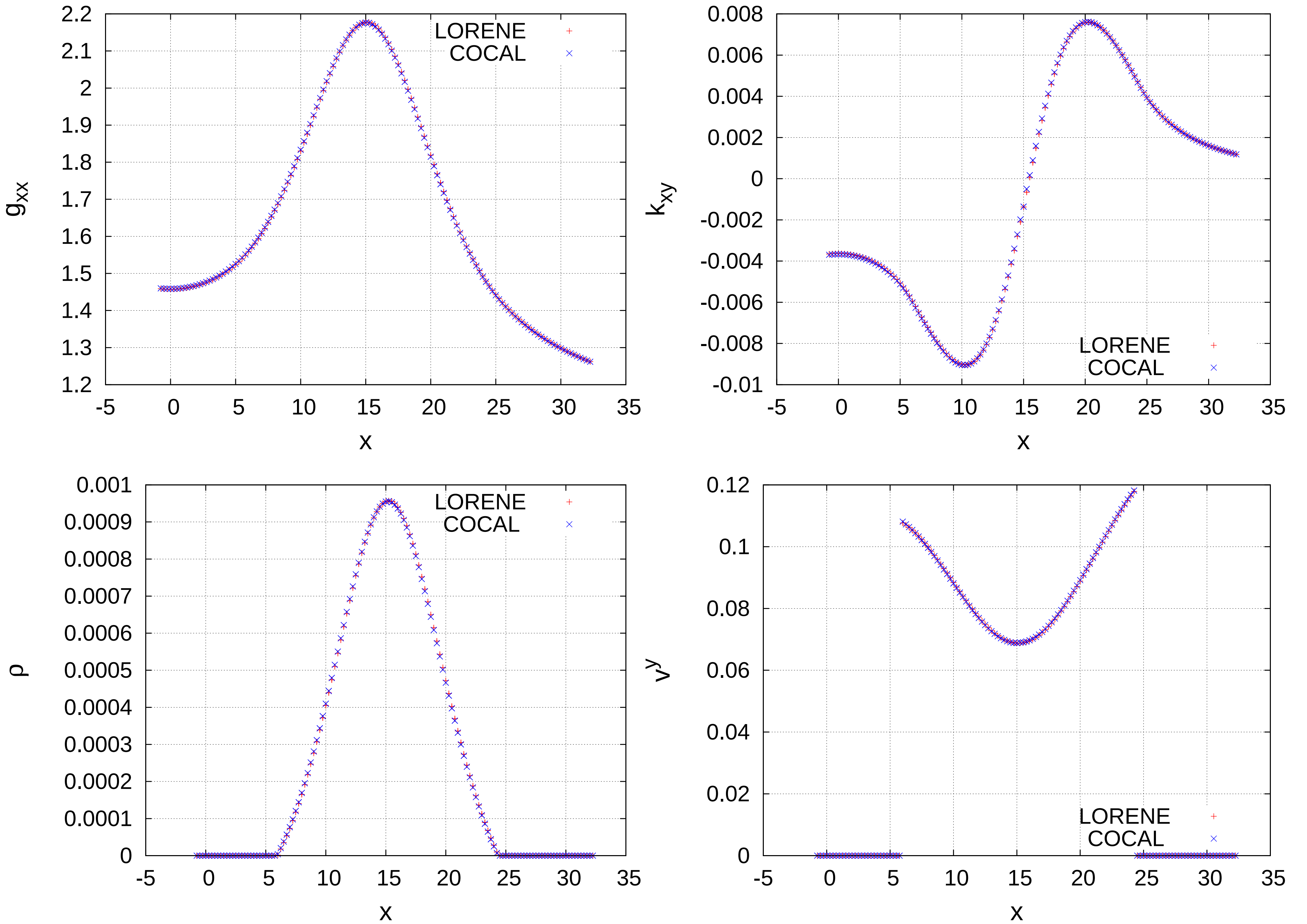




Comparison with LORENE: irrotational case



Pointwise comparison: COCAL vs LORENE on CACTUS



COCAL overview

COCAL consists of $>$ 3000 files, or

$\sim$ 6 Mb of code files, or

$\sim$ 180,000 lines of code, or

$\sim$ 2,800 pages with 64 lines per page, or

$\sim$ 9 books of 300 pages each.

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COCAL can compute

- Corotating, irrotating, spinning binary black holes in the CF approximation
- Corotating, irrotating, spinning binary neutron stars in the CF approximation
- Single rotating neutron stars with various configurations and magnetic fields, in CF or WL approximation

Your help is needed...

- Investigate more carefully the spinning binaries.
- Create asymmetric initial data.
- Reduce eccentricity of initial data.
- Try better initial data formulations.
- Parallelize
- ....