Cosmology on Simplicial Complexes

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Outline

1. Gravitation and Regge Calculus
   - Foundations of General Relativity
   - Geometric Structure of Regge Calculus
   - Time Evolution in Regge Calculus
2. The Reggecalc Library
   - Calculating the Regge Equations
   - Basic Concepts
3. Numerical Results
   - Kasner spacetime
   - Λ-vacuum spacetime

Outline

Geometry and Action Principle

- Fundamental field theories allow a description through an action principle:

$$I\left[g^{\mu\nu}\right] = \frac{1}{16\pi}\int_{\mathscr{M}} (R - 2\Lambda)\sqrt{-g}\,d^4x$$

$$\stackrel{\delta I = 0}{\Rightarrow} R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

- Lorentzian spacetime manifold $\mathscr{M}$.
- Curvature and causal structure described by $g_{\mu\nu}$.

Spacetime to Space and Time

- Solutions for the $g_{\mu\nu}$ heavily rely on symmetries imposed on spacetime $\mathscr{M}$.
- Numerically integrate more general solutions?
    - General Relativity based on the union of space and time.
    - Time evolution scheme?
- Standard way: Split Einstein field equations into spatial and temporal partial differential equations.
    - Non-linear coupled PDEs, very difficult to solve.

Principle of Equivalence

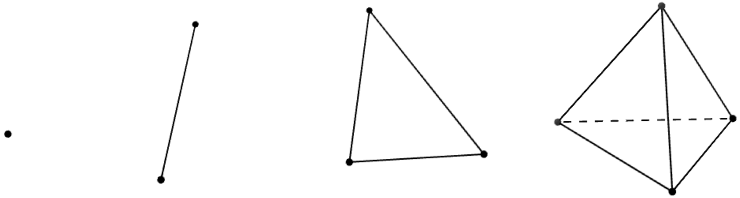
- "The laws of physics are the same in any local Lorentz frame of curved spacetime as in a global Lorentz frame of flat spacetime." (Gravitation, Misner, Thorne and Wheeler)
  - In a local region around an event spacetime looks like Minkowski space.
- Cut the universe in flat pieces and glue them together?

Outline

Regge Calculus

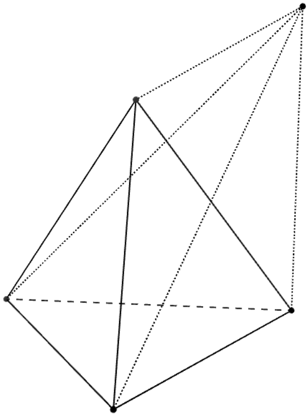
- Tullio Regge (1961):
  - Approximate the local neighborhood by finite sized blocks.
  - Interior of the blocks: flat Minkowski spacetime.
  - Fundamental degrees of freedom: edge lengths.
- Piecewise linear spacetime manifold.
  - Triangulation based on 4-simplexes.

n-Simplexes



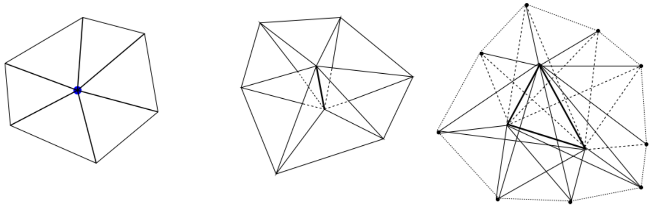
0, 1, 2 and 3-dimensional simplex.

5-Cell



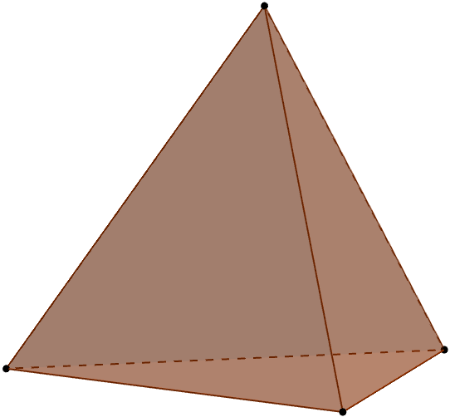
5-cell, 4-simplex or pentachoron

Hinges and Angles

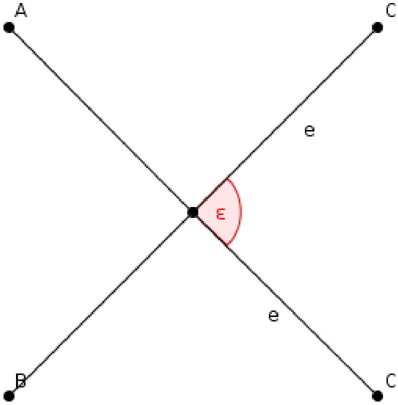


0, 1 and 2 dimensional hinges: vertex, edge and triangle. (Galassi 1992)

Defect Angle on PL 2-Sphere



Defect Angle on PL 2-Sphere



Curvature and Gravitation

- Everything known to define discrete Hilbert-Action (without cosmological constant):

$$
\begin{aligned}
I_H &= \frac{1}{16\pi}\int_{\mathcal{M}} R\sqrt{-g}\,d^4x \\
\Rightarrow I_{Regge} &= \frac{1}{16\pi}\sum_{\sigma_2} R_{\sigma_2} V_{\sigma_2} \\
&= \frac{1}{8\pi}\sum_{\sigma_2} \varepsilon_{\sigma_2} A_{\sigma_2}
\end{aligned}
$$

- T. Regge (1961): “General Relativity without Coordinates.”

Vacuum Regge Equations

- Einstein field equations follow from variation of I_H w.r.t. $g_{\mu\nu}$.
- Regge equations follow from variation w.r.t. l_j^2

$$R_j := \frac{\delta I_{Regge}}{\delta l_j^2} = \sum_{\sigma_2 \supset \sigma_1^j} \frac{\delta A_{\sigma_2}}{\delta l_j^2} \varepsilon_{\sigma_2} = 0$$

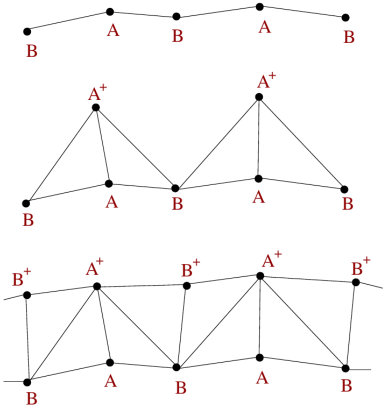
- Non-linear algebraic equations relating the edge lengths of the triangulation.

Outline

Time Evolution in Regge Calculus

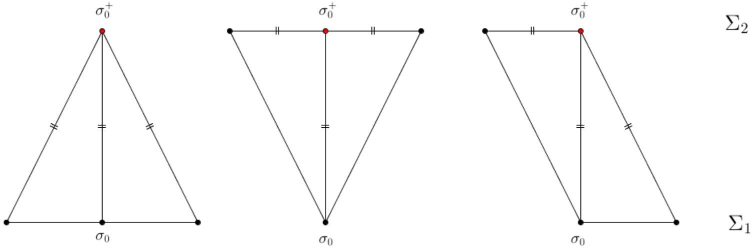
- Sorkin (1975) and Tuckey (1993):
  - Time evolution of a hypersurface by a local decoupling scheme.
- Advance vertex after vertex to the next hypersurface.
- Triangulating the intermediate region automatically by 4-simplexes.

Sorkin Scheme



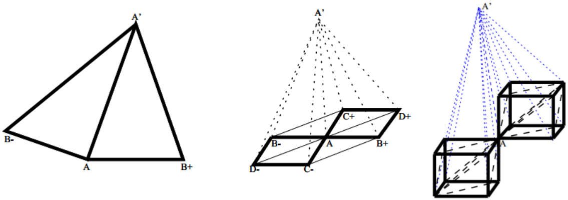
Successive steps in the Sorkin scheme. (Gentle 1999)

Sorkin Scheme



Pairs of known and unknown edges.

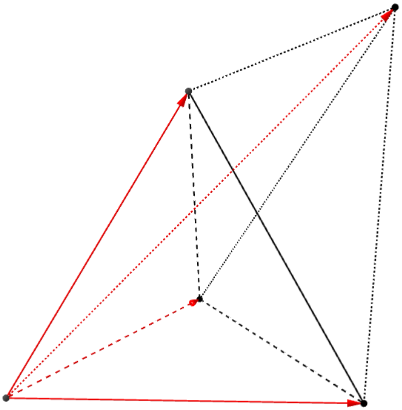
Sorkin Scheme



Sorkin scheme in one, two and three dimensions. (Galassi 1992)

Outline

The Standard Frame



$$l_{ij}^2 = g_{\mu\nu}\Delta x_{ij}^{\mu}\Delta x_{ij}^{\nu} \;\Rightarrow\; g_{\mu\nu} = \frac{1}{2}\left(l_{0\mu}^2 + l_{0\nu}^2 - l_{\mu\nu}^2\right)$$

Structure of the Equations

- Define n-volume to be positive definite

$$V_n = \frac{1}{n!}\sqrt{|g^{(n)}|} \Rightarrow A\left(l^2\right) \Rightarrow \frac{\partial A}{\partial l_j^2}$$

- Construct normal vectors to two successive tetrahedrons

$$n^\mu, m^\mu \Rightarrow \phi_{nm} \Rightarrow \varepsilon_{\sigma_2}$$

- Notion of angle depends on triangle signature:
  - Euclidean angles around timelike triangles.
  - Hyperbolic angles around spacelike triangles.

Outline

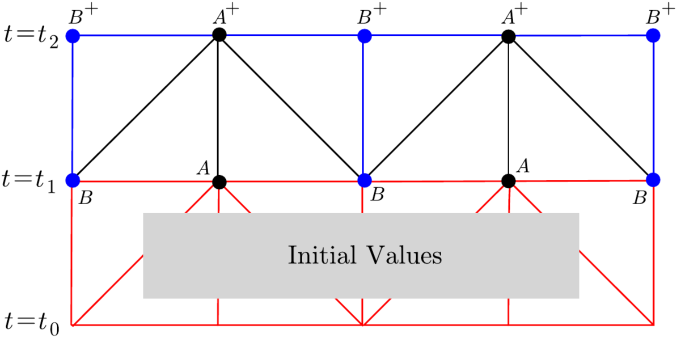
Overview

- Written in C++:
  - Split into highly flexible classes:
    triangulation, equations, time evolution
  - Allowing for arbitrary simplex weigths and additional terms.
  - Parallelized time evolution based on vertex coloring.
- Will be released as OSS under the MPL2.

Check and Initialisation

- Freely specifiable input data:
  - Hypersurface triangulation Σ of a closed PL 3-manifold.
  - Initial edge lengths.
  - Arbitrary additional terms.
- Check input hypersurface: PL closed 3-manifold?
- Generate 4-dim three-surface triangulation by Sorkin scheme.

Initial Conditions



Three-surface triangulation and initial conditions. (Peuker 2009)

Outline

Kasner Metric

- Kasner metric, solution of vacuum Einstein field equations

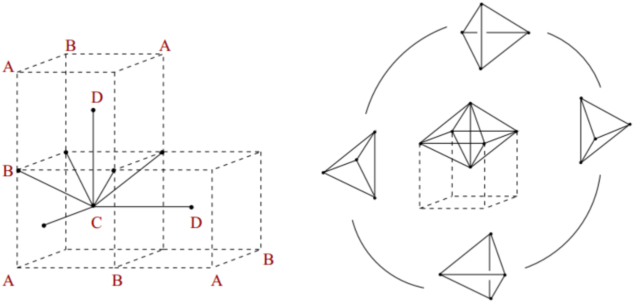
$$ds^2 = -dt^2 + t^{2p_x} dx + t^{2p_y} dy + t^{2p_z} dz$$

- With the conditions

$$\begin{aligned} p_x + p_y + p_z &= 1 \\ p_x^2 + p_y^2 + p_z^2 &= 1 \end{aligned}$$

- Homogeneous hypersurfaces,
  but anisotropic expansion/contraction

Regular Lattice



Regular lattice with vertex types A, B, C and D which can be evolved in parallel. (Gentle 1999)

Initial Conditions

- Triangulated domain: $\Sigma_0 \rightarrow \Sigma_1 \rightarrow \Sigma_2$
- Initial edge lengths from analytical solution

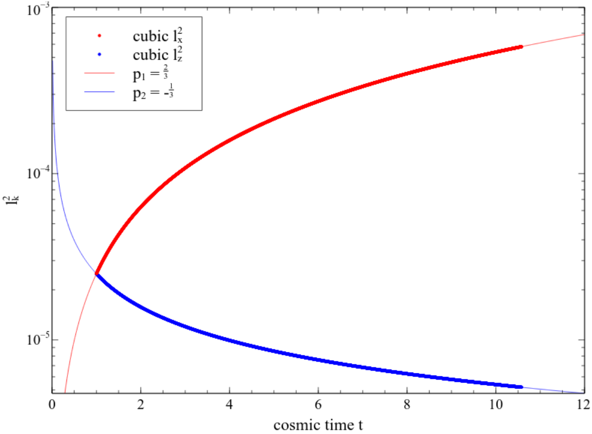
$$l^2 = \left(\int_\gamma \sqrt{g_{\mu\nu}(\lambda)\, dx^\mu dx^\nu} \right)^2$$

- Approximate by straight line $x^\mu(\lambda) = x_0^\mu + \lambda \Delta x^\mu$

$$l^2 = \left(\int_0^1 \sqrt{g_{\mu\nu}(\lambda)\, \Delta x^\mu \Delta x^\nu} d\lambda \right)^2$$

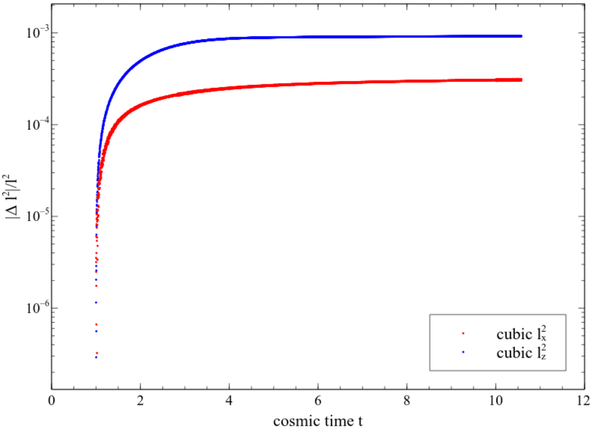
- Start at a cosmic time of $t_0 = 1$.

Time Evolution



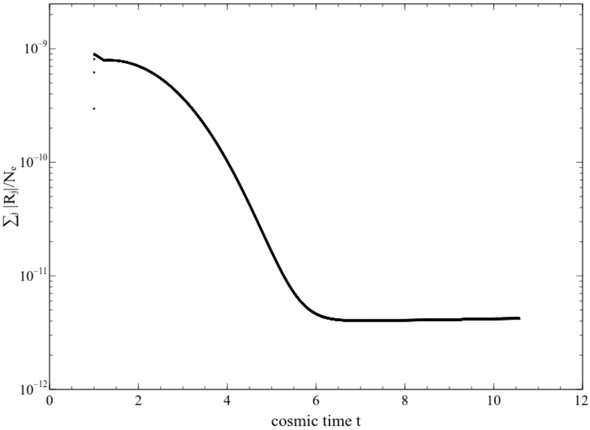
$$p_{x,y} = \frac{2}{3},\; p_z = -\frac{1}{3},\; \Delta x = 0.005,\; 10000 \text{ iterations}$$

Time Evolution



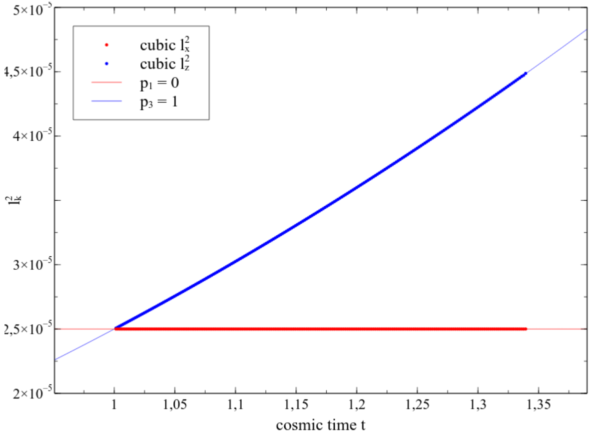
$p_{x,y} = \frac{2}{3}$, $p_z = -\frac{1}{3}$, $\Delta x = 0.005$, 10000 iterations

Time Evolution



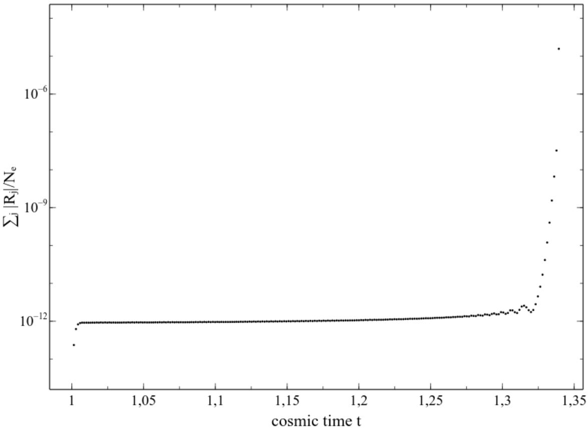
$p_{x,y} = \frac{2}{3}$, $p_z = -\frac{1}{3}$, $\Delta x = 0.005$, 10000 iterations

Time Evolution



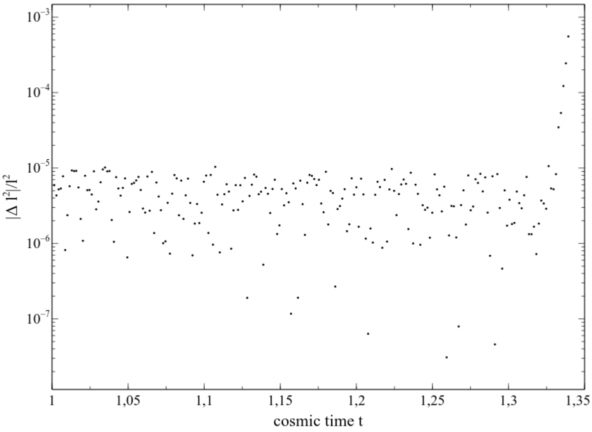
$p_{x,y} = 0, \ p_z = 1, \ \Delta x = 0.005, \ 222$ iterations

Time Evolution



$p_{x,y} = 0,\ p_z = 1,\ \Delta x = 0.005,$ 222 iterations

Time Evolution



$p_{x,y} = 0,\ p_z = 1,\ \Delta x = 0.005,$ 222 iterations

Outline

Inclusion of Λ

- Start again from Hilbert action

$$S_H = \frac{1}{2\kappa}\int R\, dV^{(4)} - \frac{1}{\kappa}\int \Lambda\, dV^{(4)}$$

- Discretize action associated with Λ

$$S_\Lambda = -\frac{\Lambda}{\kappa}\int dV^{(4)} \to -\frac{\Lambda}{\kappa}\sum_{\sigma_4} V_{\sigma_4} = -\frac{\Lambda}{\kappa}\sum_{\sigma_4}\frac{\sqrt{|g_{\sigma_4}|}}{4!}$$

Inclusion of Λ

- Regge equations with cosmological constant

$$\frac{\delta S_{R\Lambda}}{\delta l_j^2} = \frac{\delta S_R}{\delta l_j^2} + \frac{\delta S_\Lambda}{\delta l_j^2} := R_j + R_{\Lambda,j} \stackrel{!}{=} 0.$$

$$\begin{aligned} R_{\Lambda,j} &= -\frac{1}{\kappa}\frac{\Lambda}{4!}\sum_{\sigma_4 \supset \sigma_1^j} \frac{\delta\sqrt{|g|}}{\delta l_j^2} \\ &= -\frac{1}{\kappa}\frac{\Lambda}{2}\sum_{\sigma_4 \supset \sigma_1^j} V_{\sigma_4} \operatorname{tr}\left(\boldsymbol{g}^{-1} \cdot \frac{\delta \boldsymbol{g}}{\delta l_j^2}\right) \end{aligned}$$

Λ-vacuum Metric

- Flat Λ-vacuum → flat FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

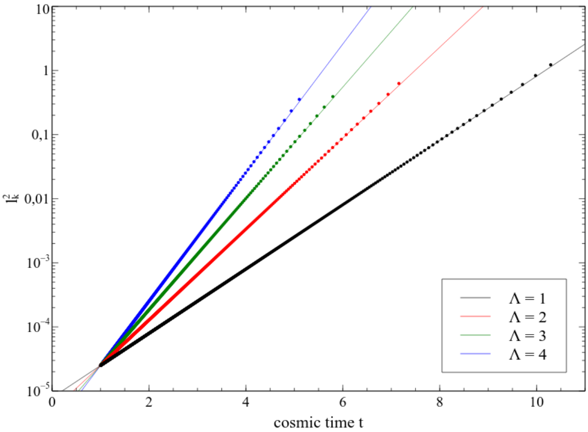
- Governed by (first) Friedman equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} = \text{const.}$$

$$\Rightarrow a(t) \propto e^{\sqrt{\frac{\Lambda}{3}}t}$$

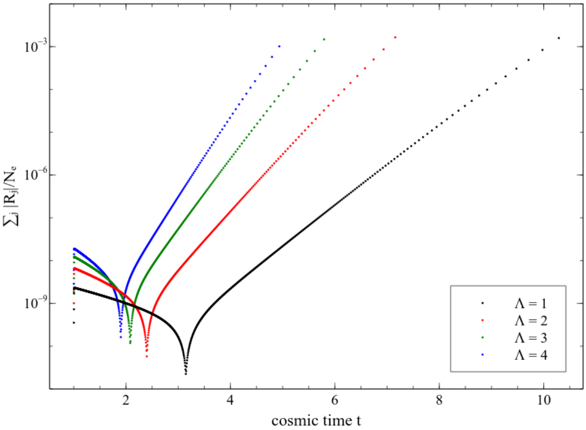
- Homogeneous flat hypersurfaces, exponentially expanding.

Time Evolution



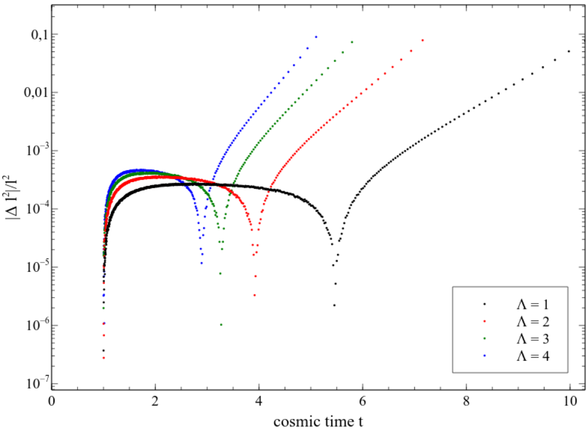
$\Lambda = 1, \Delta x = 0.005$, 1000 iterations

Time Evolution



$\Lambda = 1, \Delta x = 0.005$, 1000 iterations

Time Evolution

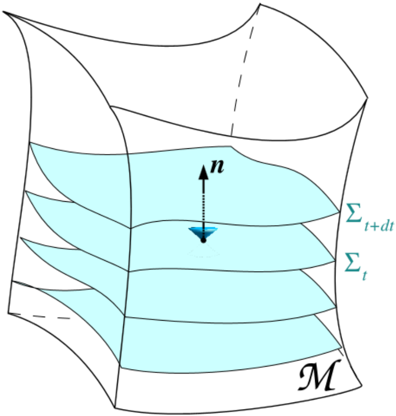


$$\Lambda \in \{1,2,3,4\}, \Delta x = 0.005$$

Conclusion

- Regge calculus is a useful method to approximate General Relativity.
  - Time evolution produces a triangulation of the spacetime.
    $\rightarrow$ fundamentally geometric, no coordinates.
  - Original method developed to couple Λ to the lattice.
    $\rightarrow$ produces the correct time evolution.
  - Next step: coupling of perfect fluid.
    $\rightarrow$ inhomogenous/anisotropic universes.
- *Reggecalc* library good starting point for further investigations.
  - Thorough stability-analysis of the involved equations needed.

Cauchy Surfaces



Cauchy surface Σ and its normal ***n*** foliating the manifold. (Gourgoulhon 2007)

3+1 Einstein equations

- Second-order non-linear PDEs (vacuum, $\Lambda = 0$):

$$\left(\frac{\partial}{\partial t} - \mathscr{L}_\beta\right)\gamma_{ij} = -2NK_{ij}$$

$$\left(\frac{\partial}{\partial t} - \mathscr{L}_\beta\right)K_{ij} \quad = \quad N\left\{R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j\right\} - D_iD_jN$$

$$R + K^2 - K_{ij}K^{ij} = 0$$

$$D_jK^j{}_i - D_iK = 0$$

Piecewise Linear Manifold

- Gravitational degrees of freedom completely described by edge lengths l_i.

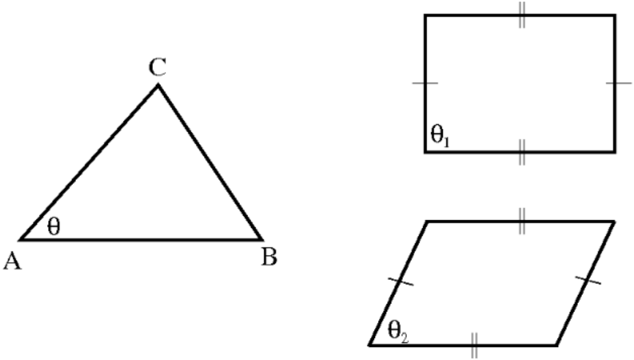
$$\{l_i\} \Longleftrightarrow g_{\mu\nu}(x) \Longleftrightarrow \boldsymbol{e}^a_\mu(\boldsymbol{x})$$

- Only n-simplexes are fully described by their l_i.

$$\frac{n(n+1)}{2}\, g_{\mu\nu} \Leftrightarrow \frac{n(n+1)}{2}\, l_i$$

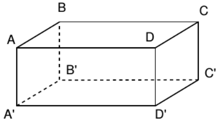
- Leads to a triangulation of spacetime by 4-simplexes.

Rigidity of Polygons

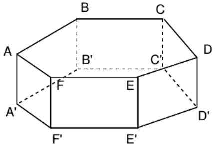


Rigidity of polygons. (Miller 2008)

Topology



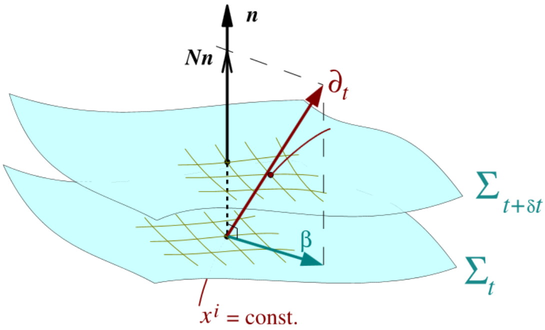
E1)	ABCD	⟷	A'B'C'D'
	ABB'A'	⟷	DCC'D'
	ADD'A'	⟷	BCC'B'
E2)	ABCD	⟷	C'D'A'B'
	ABB'A'	⟷	DCC'D'
	ADD'A'	⟷	BCC'B'
E3)	ABCD	⟷	B'C'D'A'
	ABB'A'	⟷	DCC'D'
	ADD'A'	⟷	BCC'B'
E4)	ABCD	⟷	C'D'A'B'
	ABB'A'	⟷	D'A'AD
	BCC'B'	⟷	C'D'DC



E5)	ABCDEF	⟷	C'D'E'F'A'B'
	AA'F'F	⟷	CC'D'D
	EE'D'D	⟷	AA'B'B
E6)	ABCDEF	⟷	B'C'D'E'F'A'
	AA'F'F	⟷	CC'D'D
	EE'D'D	⟷	AA'B'B

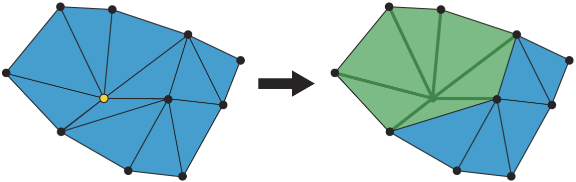
Orientable, locally Euclidean topological 3-spaces. (Lachieze-Rey and Luminet 1995)

Lapse and Shift



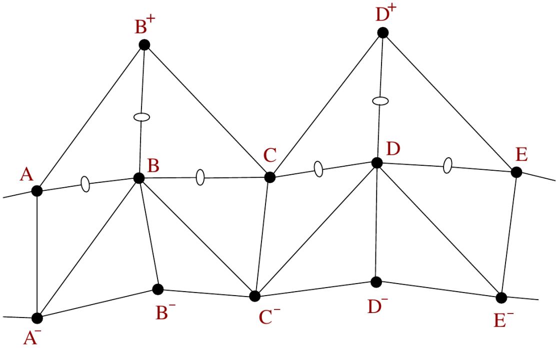
Lapse N and Shift β giving the coordinate propagation between two Cauchy surfaces. (Gourgoulhon 2007)

Star of a Vertex



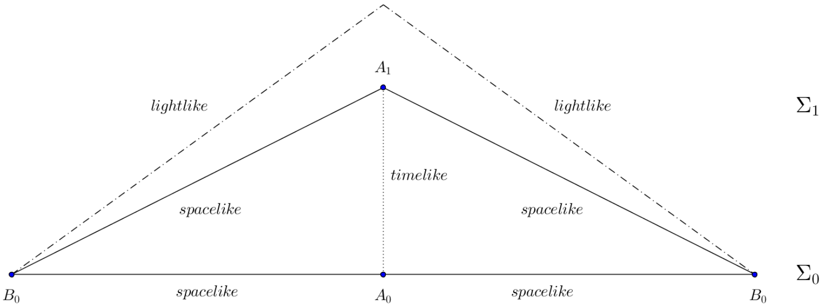
Star of a vertex in two dimensions. (Wikimedia.org)

Sorkin Scheme

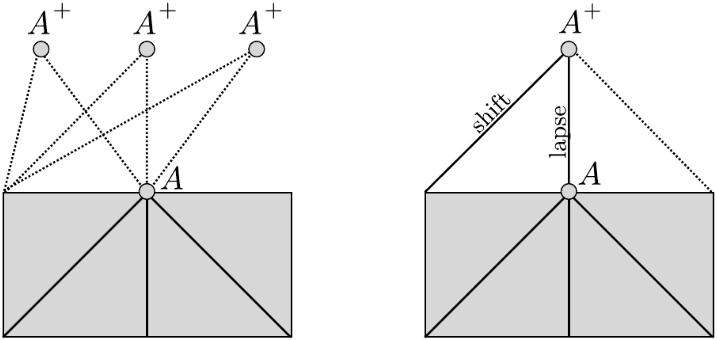


Two-surface initial condition and Regge equations. (Gentle 1999)

Time Step and Causality

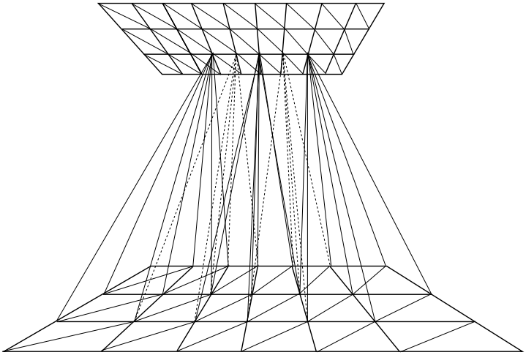


Lapse and Shift



Lapse and shift in Regge calculus. (Peuker 2009)

Sorkin Scheme in Two Dimensions



Sorkin scheme in two dimensions on a simplicial complex. (Galassi 1992)

Time Evolution

- Solve Regge equations (and additional terms) by Newton-Raphson method:

$$J_j{}^k \left(\boldsymbol{l}^2 \right) \Delta l_k^2 = -R_j \left(\boldsymbol{l}^2 \right)$$

- Jacobian $J_j{}^k$ is determined by numerical differentiation.
- Overdetermined system solved using a QR decomposition.

Shift Conditions

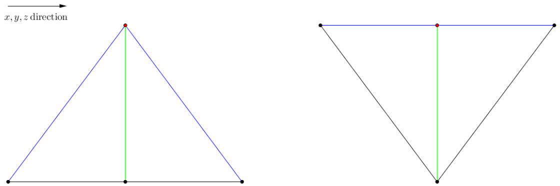
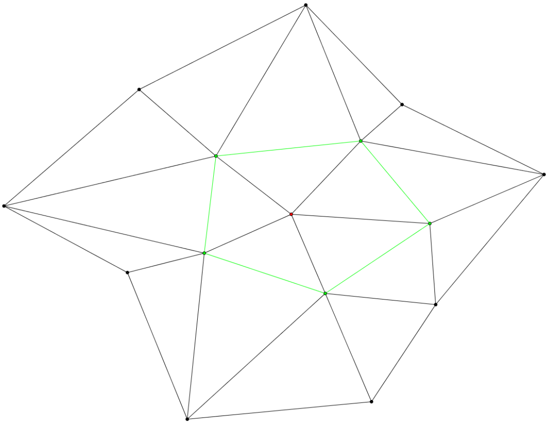
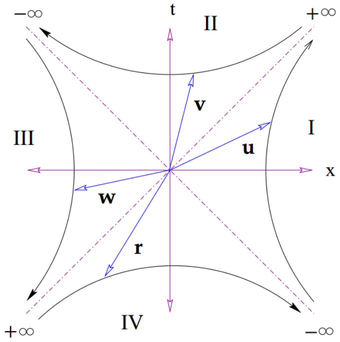


Figure : Regular lattice with vertex types A, B, C and D which can be evolved in parallel. (Gentle 1999)

Link of a Vertex



Hyperbolic Boost Angles



Hyperbolic boost angles between vectors in Minkowski plane defined on unit hyperboles. (Gentle 1999)

Historical applications

- Vacuum Regge equations first used to calculate static vacuum spacetimes
  - Schwarzschild geometry (Wong 1971)
  - Black holes with non-spherical and multiple throats (Collins and Williams 1972)
- Later on time evolution of highly symmetric spacetimes
  - RW and Tolman universes (Collins & Williams 1973/74)
  - Relativistic collapse of a spherically symmetric perfect fluid (Dubal 1989b/90)
  - Taub universe initial value problem and time evolution (Tuckey and Williams 1988)

600-Cell closed FLRW Model

- 600-cell: Triangulation of S^3.
  - 600 tetrahedrons, 120 vertices,
    12 edges meeting at every vertex.
  - Forwarded in time by 4 steps with 30 vertexes evolved in parallel.
- Action for an isolated particle of mass m:

$$I = -\int m\, ds$$

- Action of homogeneously distributed dust ("particles") on vertices:

$$I = \sum_h A_h \varepsilon_h - 8\pi \sum_i \frac{M}{120} \Delta\tau_i$$

Scale Reduction

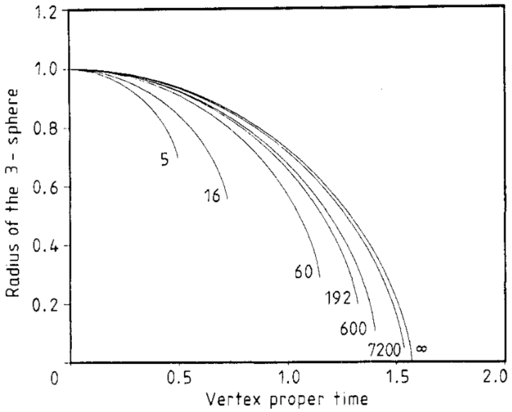


Figure : Regge approximation to closed FLRW universe through subdivisions. (Brewin 1987)

Curvature in a Simplicial Complex

- Curvature introduced by angle defects ε_h on $n-2$ dimensional hinges.
  - $n=2$: Surface tiled by triangles, ε_h at vertexes (angle).
  - $n=3$: Volume tiled by tetrahedrons, ε_h at edges (dihedral angle).
  - $n=4$: Spacetime tiled by pentachorons, ε_h at triangles (hyperdihedral angle).

Curvature by Differential Geometry

- Curvature "detection":
  - Parallel transport a unit vector $\boldsymbol{u}$ around a closed loop with enclosed area A.

$$^{(n)}R = n(n-1)\,^{(n)}K = n(n-1)\,\frac{\delta \boldsymbol{u}}{A}$$

- Simplicial complex with angle defect ε_h:
  - Parallel transport of vector around a closed loop orthogonal to the hinge h.
  - Vector comes back rotated by $\varepsilon_h = 2\pi - \sum_i \theta_i$
  - Independent of enclosed area A (conical singularity).

Curvature by Differential Geometry

- Natural choice of area where curvature is supported (Miller 1997):
  - Circumcentric dual polygon h^* (Voronoi cell).

$$^{(n)}R_h = n(n-1)\,^{(n)}K_h = n(n-1)\,\frac{\varepsilon_h}{A_{h^*}}$$

- Defines a hybrid-cell, which tiles spacetime and retains rigidity.

$$V_h^{hybrid} = \frac{2}{n(n-1)} A_h A_{h^*}$$

Bianchi Identities

- Cartan formalism also delivers ordinary Bianchi Identity:

$$\sum_{h \supset L} \hat{R}_h = 0 \Leftrightarrow R^{\alpha}{}_{\beta[\lambda\mu;\nu]}$$

- In continuum contracted Bianchi Identity:

$$R^{\alpha}{}_{\beta[\lambda\mu;\nu]} \Rightarrow \left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0$$

- Gives conservation of source $\nabla_\mu T^{\mu\nu} = 0$ and diffeomorphism invariance.

Contracted Bianchi Identity

- Cartan formalism and BBP deliver approximate Bianchi Identity

$$\sum_{L \supset V} \sum_{h \supset L} \frac{1}{2} \boldsymbol{L} \cot\left(\theta_h\right) \varepsilon_h + O\left(L^5\right) = 0$$

- Approximate diffeomorphism invariance.
  - Free choice of Lapse N and Shift β^i through evolution with error $\propto O\left(L^5\right)$.
  - Corresponds to one timelike edge and three diagonals between surfaces Σ_t.

Geodesic Deviation on Simplex

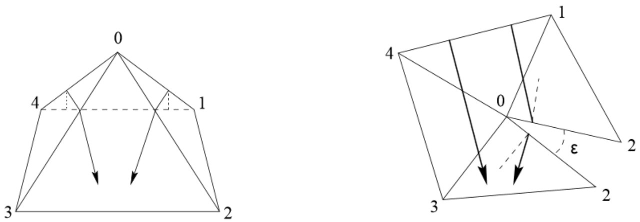


Figure : Geodesic deviation on a simplex with angle defect. (Miller 1998)

Geodesic Deviation in Simplicial Complex

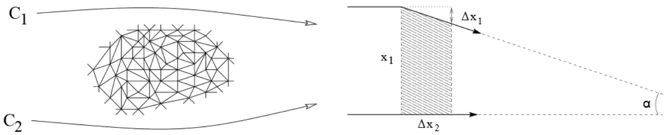


Figure : Geodesic deviation in a simplicial complex. (Miller 1998)

Initial Conditions

- Constraint equation to be fulfilled by initial conditions.
- Different approaches to split between constrained and free initial data.
  - e.g. Lichnerowicz (1944) and York (1971): Conformal decomposition of three metric.

$$\gamma = \Psi^4 \tilde{\gamma}$$

  - Further split extrinsic curvature K_{ij} into traverse and traceless parts.
  - Gentle (1998):
    York-type initial data formalism for (source-free) Regge Calculus.

Moment of Time-Symmetry

- Spacetimes with a moment of time-symmetry:

$$K_{ij} = 0 \Rightarrow R = 0 \Rightarrow \varepsilon_h = 0$$

- e.g. closed FLRW-metric.
    - Choose triangulation, e.g. regular 600-cell (homogeneous and isotropic).
    - Generate two slice initial conditions Σ_{t_0}, Σ_{t_0+dt} from Regge equations at time-symmetry.
    - Forward Σ_{t_0+dt} in time to get time evolution of closed FLRW.

$$S^3 \times \mathbb{R}$$

Discrete Exterior (Co-)Derivative

- Exterior derivative $\boldsymbol{d}$:
  k-form $\omega \xrightarrow{\boldsymbol{d}} (k+1)$-form $\eta = \boldsymbol{d}\omega$
- Exterior co-derivative δ
  k-form $\omega \xrightarrow{\delta} (k-1)$-form $\zeta = \delta\omega$
- Stokes theorem

$$\int_{\sigma^k} \boldsymbol{d}\omega = \int_{\partial\sigma^k} \omega$$

- Define discrete exterior (co-)derivative:

$$\left\langle \boldsymbol{d}\omega, \sigma^k \right\rangle = \frac{1}{|\sigma^k|} \left\langle \omega, \delta\sigma^k \right\rangle$$

- de Rham cohomology also defined for the dual lattice (Hodge dual $\star$).

Discrete Scalar Fields

- Action of a scalar field:

$$I_S\left[\phi,\bar{\phi}\right]=\int\frac{1}{2}\left(\partial^{\mu}\phi\partial_{\mu}\bar{\phi}-m^2\bar{\phi}\phi\right)d^4x=\frac{1}{2}\left(\boldsymbol{d}\phi,\boldsymbol{d}\bar{\phi}\right)-\frac{m^2}{2}\left(\phi,\bar{\phi}\right)$$

$$(\omega,\eta):=\int\omega\wedge\star\eta\rightarrow\sum_{\sigma^k}\langle\omega,\eta\rangle V^n_{\sigma^k}$$

- Projecting the 0- and 1-forms on the lattice gives

$$\langle\phi,v\rangle=\phi\left(v\right)$$

$$\langle\boldsymbol{d}\phi,L\rangle=\sum_{v\subset L}\frac{1}{|L|}\langle\phi,v\rangle=\frac{\phi\left(v+\boldsymbol{L}\right)-\phi\left(v\right)}{|L|}$$

Regge and Scalar Action

$$I_S\left[\phi,\bar{\phi}\right]=\sum_L \frac{1}{2}\frac{\phi\left(v+\mathbf{L}\right)-\phi\left(v\right)}{|L|}\frac{\bar{\phi}\left(v+\mathbf{L}\right)-\bar{\phi}\left(v\right)}{|L|}\frac{1}{4}|L|V_{L^*}$$
$$-\frac{m^2}{2}\sum_v \phi\left(v\right)\bar{\phi}\left(v\right)V_{v^*}$$

- Minimal coupling to Regge action I_R and thus Gravitation

$$I=I_R+I_S \overset{\delta I=0}{\Rightarrow} \frac{\delta I_R}{\delta L}+\frac{\delta I_S}{\delta L}=0$$

- One system of equations per edge L.