An Exact Approach to Head-On Collisions of Black Holes

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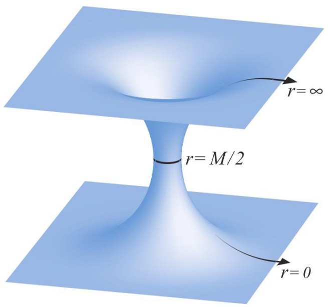
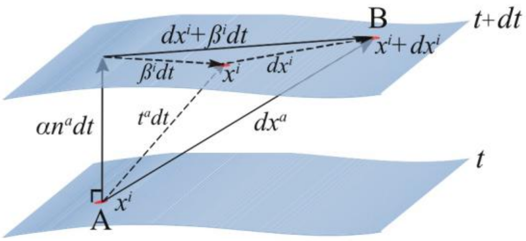
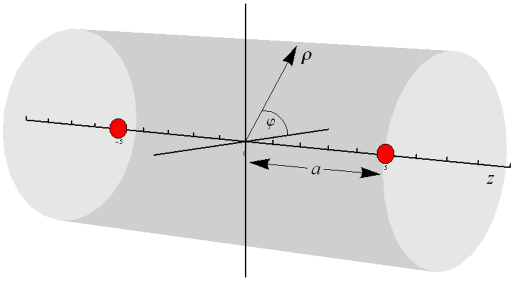
Outline of the talk

- General relativity
  - Schwarzschild black holes
- 3+1 splitting of spacetime
  - Weyl curvature
- Local rotational symmetry
  - Evolution equations
- Results
  - Weyl curvature
  - Infall times







Schwarzschild Black Holes

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{R}\right)^{-1}\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2$$

- Ingredients
  - Spherical symmetry
  - Static and stationary
  - Vacuum field equations

    $$R_{ab} = 0$$

  - Asymptotic flatness
- Properties
  - Asymptotically Minkowski
  - Event horizon, coordinate singularity at $R = 2M$
  - Curvature Singularity for

    $$R \to 0$$

  - Birkhoff's theorem

Infall Time

Distant observer:
(coordinate time)

$$\mathrm{d}s^2 = 0$$

$$\frac{\mathrm{d}t}{\mathrm{d}R} = \pm\left(1 - \frac{2M}{R}\right)^{-1}$$

$$t - t_0 \sim \ln\left(R - R_S\right) \overset{R \to R_S}{\longrightarrow} \infty$$

- Photon <u>never</u> reaches event horizon !

Infalling observer:
(proper time)

$$\mathrm{d}s = -\mathrm{d}\tau$$

$$\dot{R}^2 = \gamma^2 - \left(1 - \frac{2M}{R}\right)$$

$$\gamma = \pm\sqrt{1 - \frac{2M}{R_0}}$$

$$R = \frac{R_0}{2}(1 + \cos\eta)$$

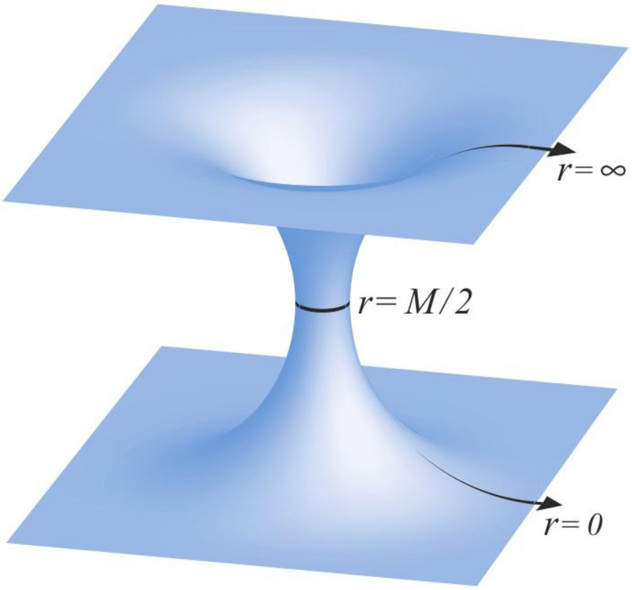
$$\tau = \frac{R_0}{2}\left(\frac{R_0}{2M}\right)^{1/2}(\eta + \sin\eta)$$

- <u>Finite</u> infall time !

Isotropic Coordinates

$$ds^2 = -\frac{\left(1-\frac{M}{2r}\right)^2}{\left(1+\frac{M}{2r}\right)^2}dt^2 + \left(1+\frac{M}{2r}\right)^4\left(dr^2 + r^2 d\Omega^2\right)$$

- Resolves coordinate singularity !
- Event horizon still problematic
- Covers only exterior of BH



Einstein-Rosen bridge

Interaction of Two Black Holes

- Geometrostatics

Bare masses: $m_i = M_i + \sum_{j\neq i} \frac{M_i M_j}{2r_{ij}}$

Total energy: $M_\Sigma = \sum_{i=1}^{N} M_i \overset{N=2}{=} M_1 + M_2$

$$\Rightarrow M_\Sigma = \sum_{i=1}^{N} m_i - \sum_{i=1}^{N}\sum_{j\neq i} \frac{M_i M_j}{2r_{ij}}$$

$$\equiv \sum_{i=1}^{N} m_i + m_{\mathrm{int}}$$

- Area Theorem

$\delta A \geq 0$ **A black hole's surface area cannot decrease**

$$A \sim R^2 \sim m^2$$

$$m_f^2 \geq m_1^2 + m_2^2$$

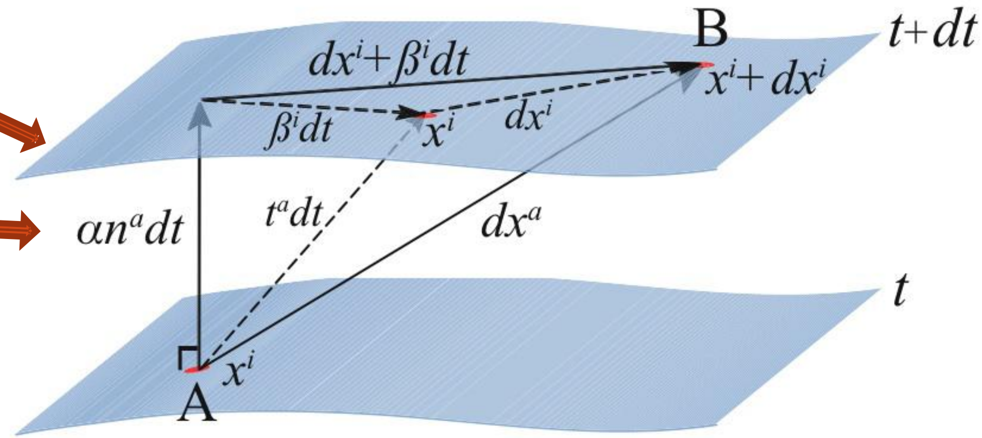
$\Delta E = 1 - \frac{m_f}{M_\Sigma}$ **Dissipated energy**

$\Delta E_{\mathrm{max}} \sim 29\%$ **Upper limit**

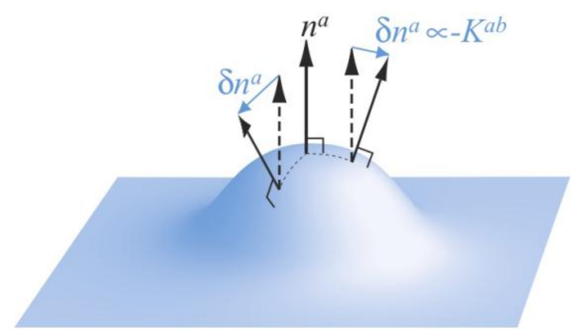
$\Delta E = 0.2\%$ **Anninos et al. (1993)**

Decomposition of Space and Time

- Spatial hypersurfaces
- Timelike unit vector
- Lapse function α
- Shift vector β^i
- Intrinsic curvature $\gamma_{ab} = g_{ab} + n_a n_b$
- Extrinsic curvature $K_{ab} \equiv -\gamma_a{}^c \gamma_b{}^d \nabla_c n_d$



$$g_{ab} = \begin{pmatrix} -\alpha^2 + \beta_l \beta^l & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}$$



Conformal Flatness

- Gauss' equation
- Einstein field eq.

$$^{(3)}R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{cd} = \gamma^p{}_a\gamma^q{}_b\gamma^r{}_c\gamma^s{}_d\,{}^{(4)}R_{pqrs}$$

Hamiltonian constraint

$$^{(3)}R + K^2 - K_{ab}K^{ab} = 16\pi\rho$$

- Conformal metric

$$\gamma_{ij} = \psi^4\bar{\gamma}_{ij} \qquad \bar{\gamma}_{ij} \equiv \eta_{ij}$$

$$8\bar{D}^2\psi - \psi\,{}^{(3)}\bar{R} - \psi^5K^2 + \psi^5K_{ij}K^{ij} = -16\pi\psi^5\rho$$

- Time symmetry
- Spherical symmetry
- Vacuum solution

$$\bar{D}^2\psi = 0 \quad \Rightarrow \quad \boxed{\psi = 1 + \sum_n \frac{M_n}{2r_n}}$$

Weyl Curvature

Riemann tensor $R^a{}_{bcd}$

→ **Trace: Ricci tensor** R_{ab}

→ **Trace-free part: Weyl conformal tensor** C_{abcd}

↓

„Electric" part: $E_{ab} = C_{acbd} n^c n^d$

Relativistic hydrodynamics: $\nabla_a n_b = -n_a \dot{n}_b + \theta_{ab} = -n_a \dot{n}_b + \sigma_{ab} + \frac{1}{3}\Theta \gamma_{ab} - \omega_{ab}$

Orthonormal frame: $\{e_\mu{}^a\}$ $\quad n^a = e_0{}^a$ $\quad T^{\mu\nu\ldots}{}_{\rho\sigma\ldots} = e^\mu{}_a e^\nu{}_b \ldots e^c{}_\rho e^d{}_\nu \ldots T^{ab\ldots}{}_{cd\ldots}$

$$E_{\alpha\beta} = \frac{1}{3}\Theta\sigma_{\alpha\beta} - \sigma_{\alpha\gamma}\sigma^\gamma{}_\beta - \omega_\alpha\omega_\beta - 2\omega_{(\alpha}\Omega_{\beta)} + \frac{1}{3}\delta_{\alpha\beta}[2\sigma^2 + \omega^2 + 2\omega_\gamma\Omega^\gamma] + S_{\alpha\beta}$$

Time symmetry →

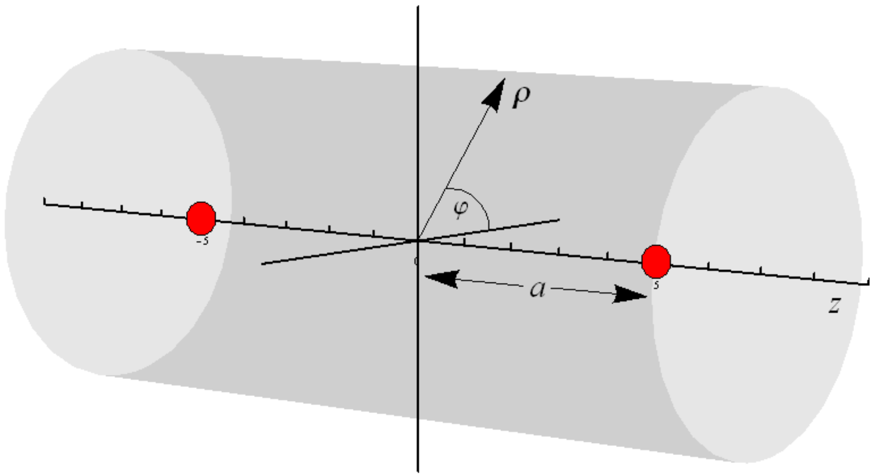
$$\boxed{E_{\alpha\beta} = {}^{(3)}R_{\alpha\beta}}$$

Local Rotational Symmetry

- Choose cylindrical geometry
- Non-vanishing variables on LRS curve:

$$\Theta, \quad \sigma_+, \quad E_+, \quad H_+$$

$$E_+ \equiv -\frac{3}{2} E_{11}$$



$$\mathrm{d}l^2 = \psi^4(\rho, z)\, \eta_{ij}(\rho, z, \phi)\, \mathrm{d}x^i \mathrm{d}x^j$$

$$= \left(1 + \frac{M_1}{2\sqrt{\rho^2 + (z+a)^2}} + \frac{M_2}{2\sqrt{\rho^2 + (z-a)^2}}\right)^4 (\mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2 \mathrm{d}\phi^2)$$

Evolution Equations

Definition of the expansion tensor

$$\theta_{11} = \frac{1}{3}(\Theta - 2\sigma_+)$$

$$\theta_{22} = \frac{1}{3}(\Theta + \sigma_+)$$

ADM equation

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 2\sigma^2 + 2\omega^2$$

and

$$\dot{\theta}_{11} + \theta_{11}^2 = \frac{2}{3}E_+$$

$$\dot{\theta}_{22} + \theta_{22}^2 = -\frac{1}{3}E_+$$

$$\dot{E}_+ + 3\theta_{22}E_+ = 0$$

Scale factor

$$\theta_{22} \equiv \partial_\tau \ln a_\perp$$

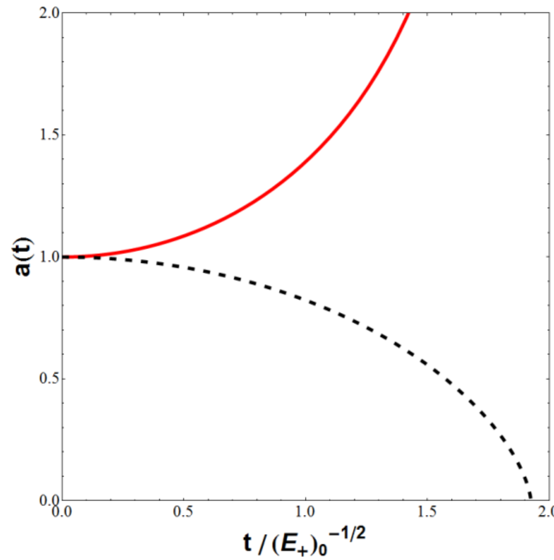
Curvature

$$E_+ = \frac{(E_+)_0}{a_\perp^3}$$

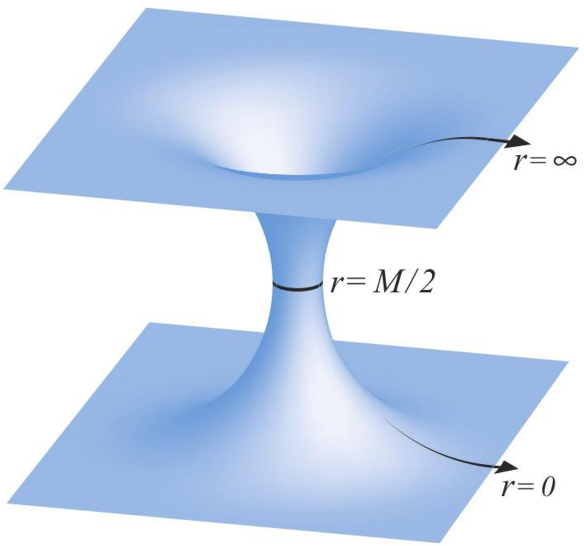
Infall time

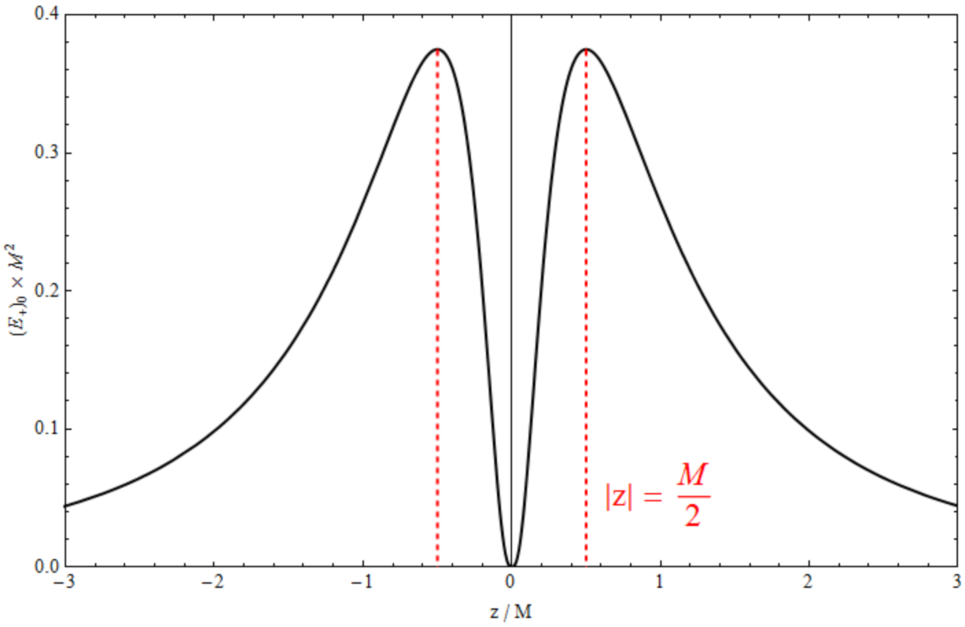
$$a_\perp = \cos^2\eta,$$

$$\tau - \tau_0 = \frac{1}{\sqrt{\frac{2}{3}(E_+)_0}}\left(\eta + \frac{1}{2}\sin(2\eta)\right)$$



Weyl Curvature for 1 BH





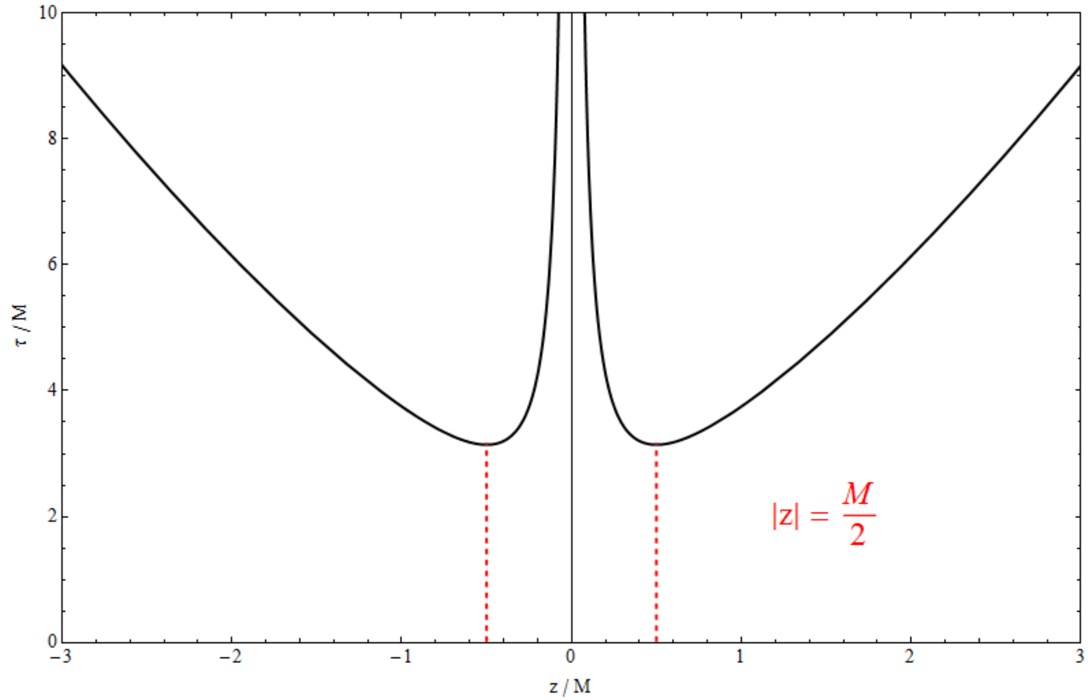
- Asymptotically flat
- Maximum = Schwarzschild radius
- „Parallel universe" for $-M/2 \leq z \leq M/2$

$$(E_+)_0 = \frac{192\,|x|^3}{M^2\,(1+2\,|x|)^6}$$

Infall Time for 1 BH

- Observers falling from position „z“ in finite time
- Minimum = Schwarzschild Radius
- Falling from „parallel universe“ for
  $-M/2 \leq z \leq M/2$
- Isotropic coordinates $\approx$ Schwarzschild coord. at infinity

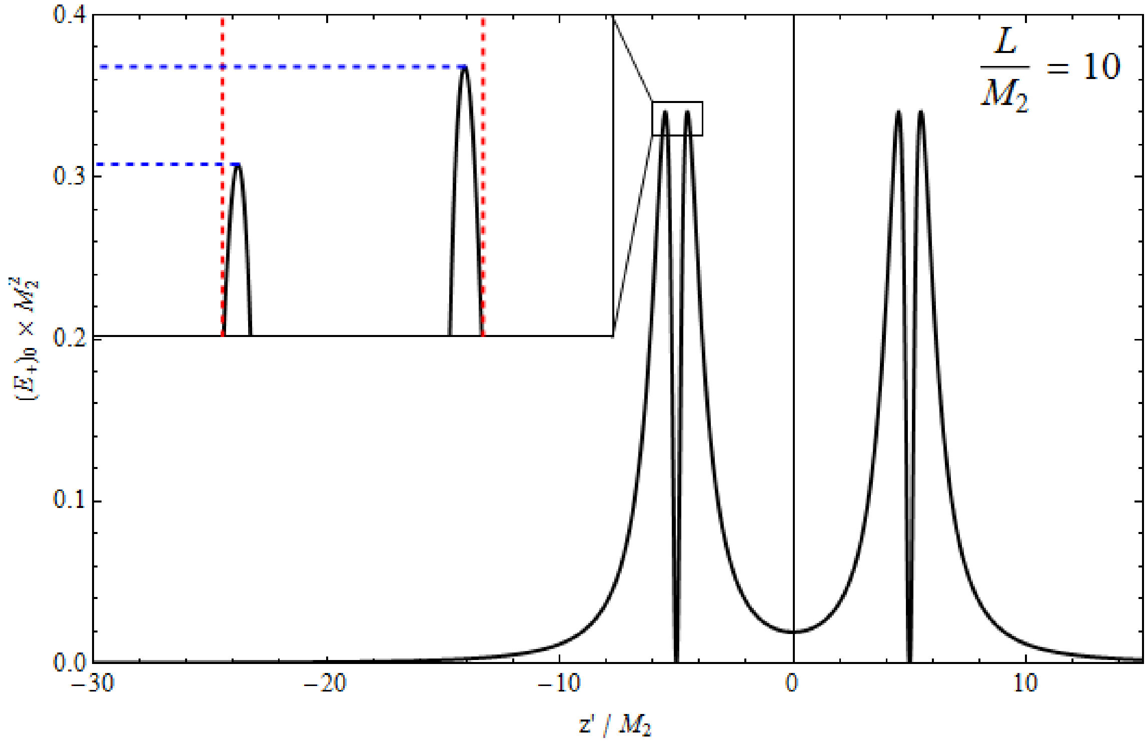
$\eta = \pi/2 \text{ and } t_0 \equiv 0$



$$\tau_{1BH} = \frac{\pi}{2} \frac{\left(\frac{M}{2} + |z|\right)^3}{\sqrt{2M\,|z|^3}} = \frac{\pi}{2} \frac{z^{3/2}}{\sqrt{2M}} + \mathcal{O}\left(z^{1/2}\right)$$

Weyl Curvature for 2 BHs

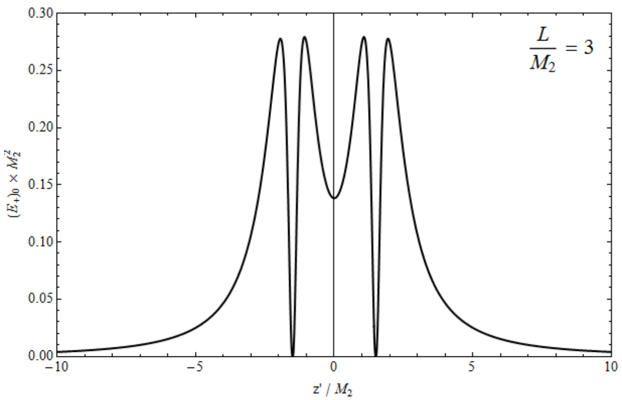
- „Superposition" of two BHs (equal mass)
- Event horizons smaller due to interaction energy
- Very small difference in the peak's heigth

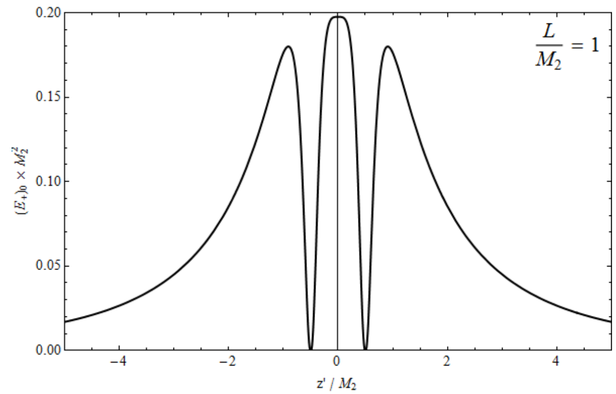


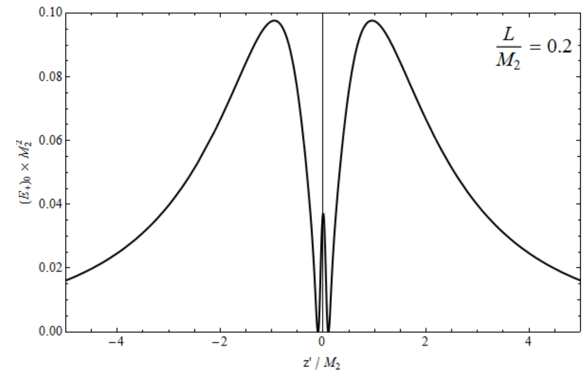
$$(E_+)_0 = \frac{24\bar{L}_+ \left(\bar{L}^2 - 4x'^2\right)^2}{M_2^2 \left(\bar{L}_+ + \bar{L}_- \left(\bar{L}_+ + 1\right)\right)^6} \Big[4\bar{L}\left(x'\bar{L}_-\bar{L}_+ - 8x'^3\right) + 4x'^2\left(\bar{L}_-\bar{L}_+ + 4x'^2\right)$$

$$+ \bar{L}^2\left(\bar{L}_-\left(\bar{L}_+ + 4\right) + 24x'^2\right) - 8x'\bar{L}^3 + \bar{L}^4\Big]$$

Variation of Separation







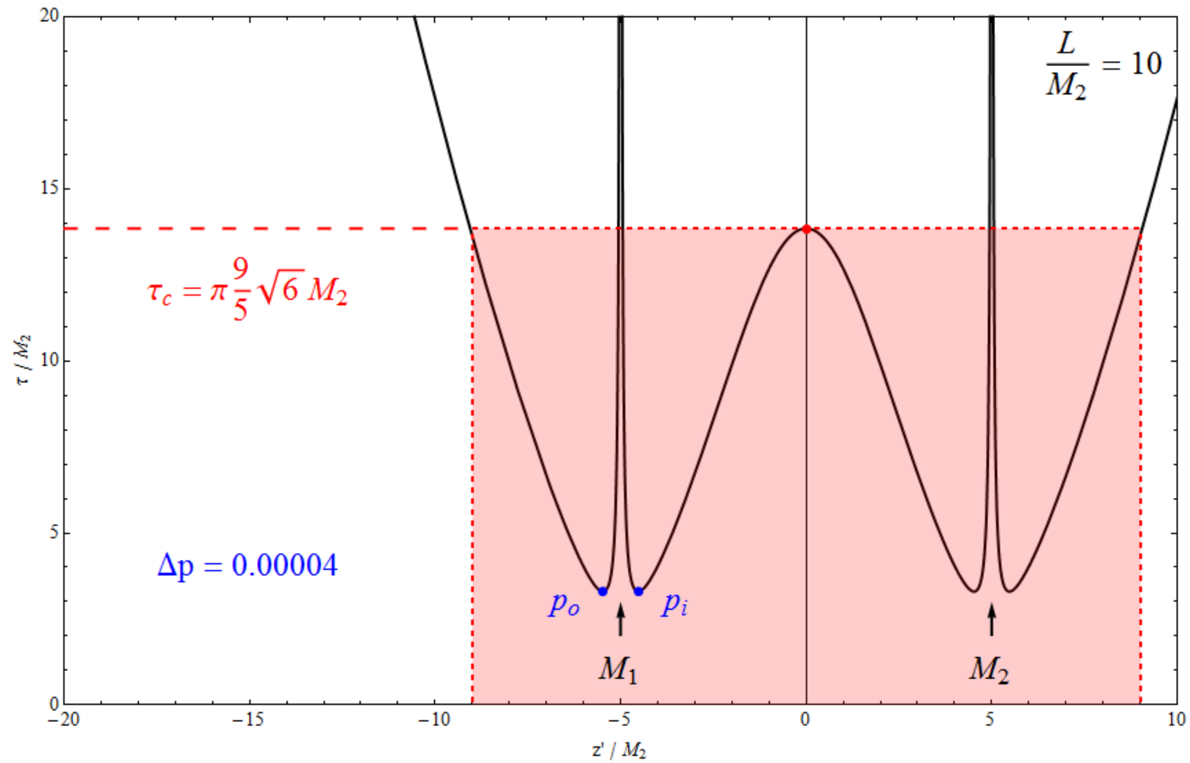
- Initial hypersurfaces, **no dynamics !**
- Difference in peak‘s height becomes larger
- Middle minimal peak rises

- Merging of peaks = Merging of event horizons ?
- Merging for **exactly L=1**
- Should merge for **L< 1** due to interaction energy

- Middle peak shrinks
- Recover single black hole curvature for L⟶ 0
- No enclosing apparent horizon visible

Infall Time for 2 BHs

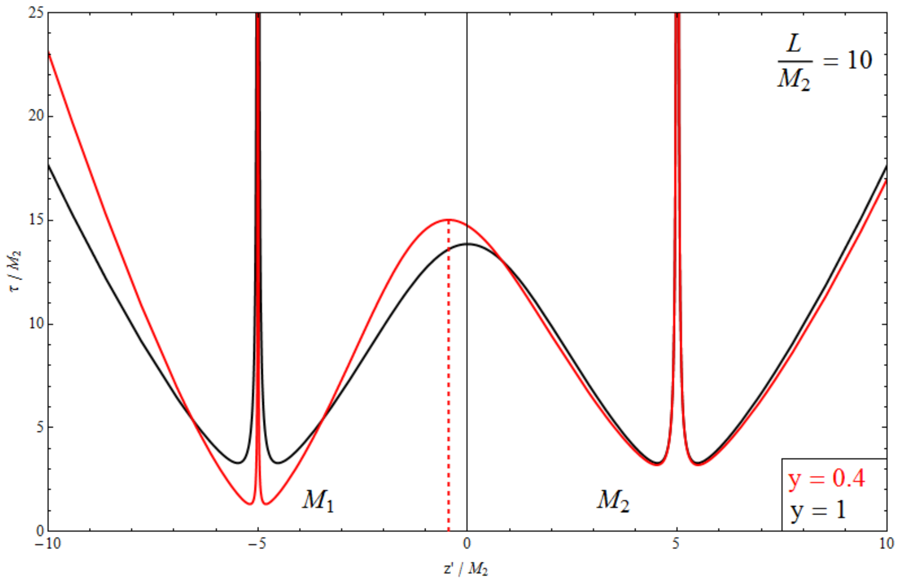
- High symmetry allows to study the merger at z‘=0
- Red box: Infall into separated BHs
- Fully dynamic motions of the BHs and the infalling observers!

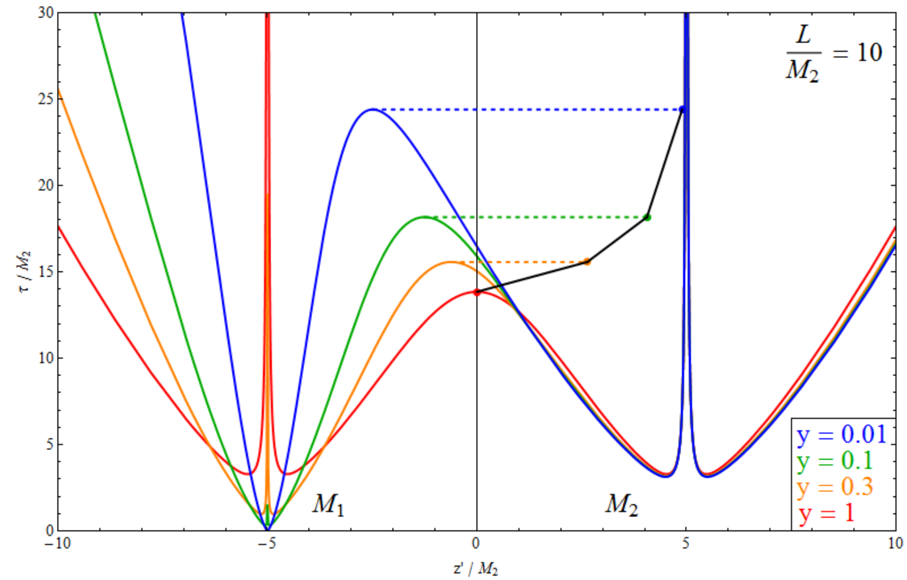


$$\tau_{2BH} = \frac{\pi}{2}\frac{z'^{3/2}}{\sqrt{2(M_1+M_2)}} + \mathcal{O}\left(z'^{1/2}\right)$$

First order: single black hole

Unequal Masses





- Scaling Problem:

$$y \equiv \frac{M_1}{M_2}$$

$$R_S \sim M$$

$$\tau \sim 1/\sqrt{M}$$

- Broken symmetry:
  - Maximum rises and shifts towards smaller BH
  - Collision coordinates not computable analytically

- Center of mass (Newtonian):

$$z'_{\text{com}} = \frac{\bar{L}}{2} \frac{y-1}{y+1+\frac{y}{L}} M_2$$

Summary and Conclusion

- We derived:
  - Evolution equations along LRS curves

$$\tau - \tau_0 = \frac{1}{\sqrt{\frac{2}{3}(E_+)_0}}\left(\eta + \frac{1}{2}\sin(2\eta)\right)$$

  - Weyl curvature and infall times for
    - Single BH
    - Equal mass 2 BH
    - Unequal mass 2BH
  - Collission times for
    - Equal mass 2BH

- Open questions:
  - Interpretation of the merging process
  - Analytic collision coordinates for unequal mass case
  - Dissipated energy, final black hole mass?