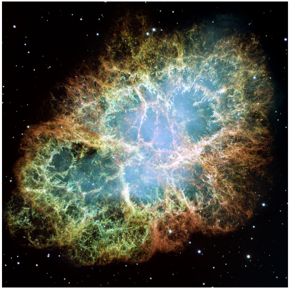
Universal Relations for the Moment of Inertia in Relativistic Stars

Cosima Breu

Goethe Universität Frankfurt am Main

Astro Coffee

Motivation

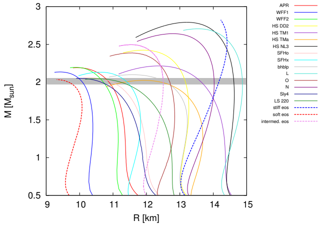


Crab-nebula (de.wikipedia.org/wiki/Krebsnebel)

- neutron stars as laboratories for unknown nuclear physics at supra-nuclear energy densities
- neutron star properties sensitively dependent on the modeling EOS
- gravitational field determined by mass, radius and higher multipole moments
- approximately universal relations between certain quantities
- constraints on EOS and quantities which are not directly observable

The Tolman-Oppenheimer-Volkoff Equations

- first solution of Einstein's equations for non-vacuum spacetimes



- Einstein equations: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$
- Metric of a spherically symmetric matter distribution:

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin\theta^2 d\phi^2)$$
$$e^{-\lambda(r)} = 1 - \frac{2M(r)}{r}$$

- Energy-momentum tensor of a perfect fluid: $T_{\mu\nu} = (e+p) u_\mu u_\nu + p g_{\mu\nu}$
- EOS needed to close system of equations

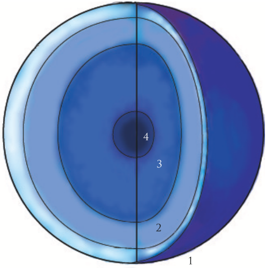
TOV-equations

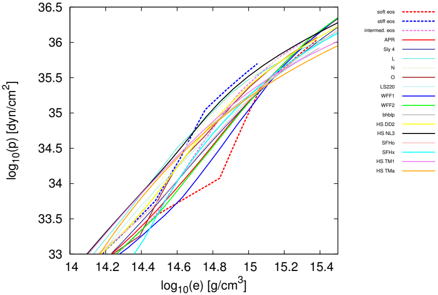
$$\frac{dM}{dr} = 4\pi r^2 e$$
$$\frac{dp}{dr} = -\frac{(e+p)(M + 4\pi r^3 p)}{r(r-2M)}$$
$$\frac{d\nu}{dr} = -\frac{2}{(e+p)} \frac{dp}{dr}$$

UNIVERSITÄT FRANKFURT AM MAIN

The Equation of State

- One-parameter EOS assumed: $p = p(\rho)$
- neutrons in β-equilibrium with protons and electrons (myons and hyperons at higher densities)
- $T = 0$

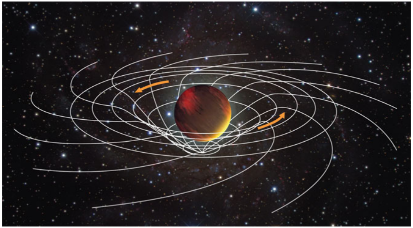




1. solid outer crust (e, Z)
2. inner crust (e, Z, n)
3. outer core (n, Z, e, μ)
4. inner core (here be monsters): large uncertainties at high densities

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

The Slow Rotation Approximation



www.vice.com/read/the-learning-corner-805-v18n5

Metric of a stationary, axisymmetric system:

$$ds^2 - \mathrm{e}^{\nu(r)} dt^2 + \mathrm{e}^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin\theta^2 (d\phi - L dt)^2)$$

Expansion of $L(r,\theta)$: $L(r,\theta) = \omega(r,\theta) + O(\omega^3)$

- local inertial frames are dragged along by the rotating fluid
- expansion until first order in the angular velocity

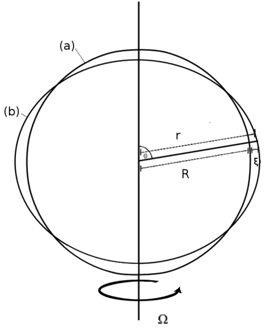
Scale-invariant differential equation for the angular velocity relative to the local inertial frame:

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4 j \frac{\bar{\omega}}{dr}\right) + \frac{4}{r}\frac{dj}{dr}\bar{\omega} = 0, \qquad j(r) = \exp\left[(-\nu + \lambda)/2\right]$$

Coordinate angular velocity: $\bar{\omega} = \Omega - \omega$, outside: $\bar{\omega} = \Omega - \frac{2J}{r^3}$

The Hartle-Thorne Perturbation Method

- perturbative expansion up to second order in the angular velocity
- Second order: changes in pressure and energy density



Expansion of the metric in spherical harmonics

$$ds^2 = -e^{\nu}(1+2h)dt^2 + e^{\lambda}[1+2m/(r-2M)]dr^2 + r^2(1+2k)[d\theta^2 + \sin\theta^2(d\phi - \omega dt)^2] + \mathcal{O}(\Omega^3)$$

$$h(r,\theta) = h_0(r) + h_2(r)P_2(\theta) + \ldots$$
$$k(r,\theta) = k_0(r) + k_2(r)P_2(\theta) + \ldots$$
$$m(r,\theta) = m_0(r) + m_2(r)P_2(\theta) + \ldots$$
$$P_2 = (3\cos\theta^2 - 1)/2$$

Coordinate system

$$r = R + \xi(R,\theta) + \mathcal{O}(\Omega^4)$$

Numerical Setup

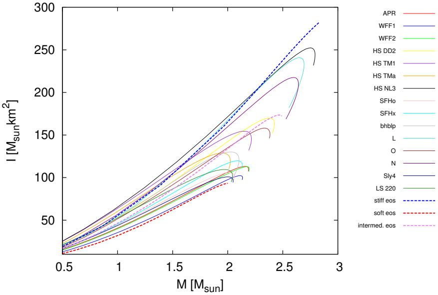
Slow Rotation:

- calculate the distribution of e,p and the gravitational field for a static configuration
- perturbations calculated retaining only first and second order terms
- equilibrium equations become a set of ordinary differential equations
- Fourth-order Runge-Kutta algorithm
- boundary conditions have to ensure metric continuity and differentiability

Rapid Rotation:

- RNS-code for uniformly rotating stars
- solve hydrostatic and Einstein's field equations for rigidly rotating, stationary and axisymmetric mass distributions
- KEH-scheme: elliptic-type field equations converted into integral equations

The Moment of Inertia



The moment of inertia

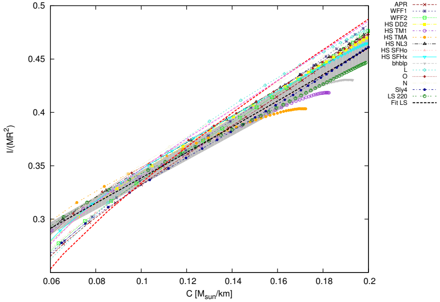
$$J = \int T^t_\phi \sqrt{-g}\, d^3x$$

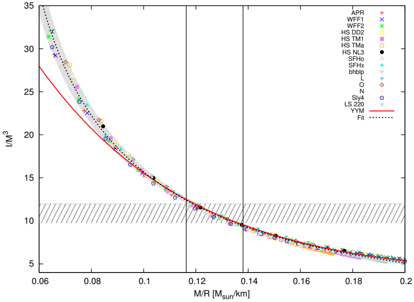
$$I(M,\nu) = J/\Omega$$

$$= \frac{8\pi}{3} \int_0^R r^4 \frac{(e+p)}{1-2M(r)/r} j(r) \frac{\bar{\omega}}{\Omega} dr$$

- Slow Rotation: Angular momentum linearly related to moment of inertia

I-$\mathscr{C}$-Relation





Lattimer and Schutz 2005

$$I/(MR^2) = a + b\frac{M}{R} + c\left(\frac{M}{R}\right)^4$$

New Fit

$$\frac{I}{M^3} = \frac{a}{\mathscr{C}^2} + \frac{b}{\mathscr{C}^3} + \frac{c}{\mathscr{C}^4}$$

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

The Quadrupole Moment

- deviation of the gravitational field away from spherical symmetry
- extracted from the asymptotic expansion of the metric functions at large r

Quadrupole moment

$$Q^{(rot)} = -\frac{J^2}{M} - \frac{8}{5} K M^3$$
$$\bar{Q} = QM/J^2$$

- $\bar{Q}$ approaches 1 for a Kerr black hole

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

The Tidal Love Number

- deformability due to tidal forces
- ratio between tidally introduced quadrupole moment and tidal field due to a companion NS

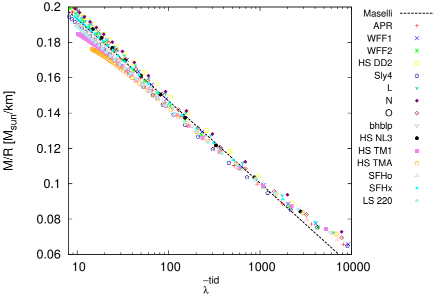
Tidal Love number

$$-\frac{1-g_{tt}}{2} = -\frac{M}{r} - \frac{3Q_{ij}}{2r^3}n^i n^j + \frac{\varepsilon_{ij}}{2}r^2 n^i n^j$$

$$\lambda^{(tid)} = \frac{Q^{(tid)}}{\varepsilon^{(tid)}}$$

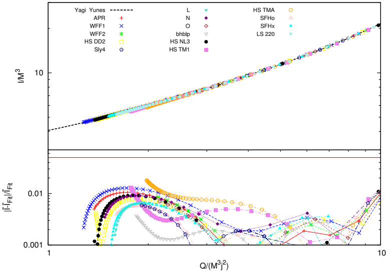
- Fit: $\mathscr{C} = a + b\ln\bar{\lambda} + c(\ln\bar{\lambda})^2$

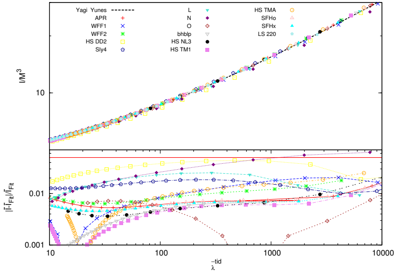
- detection of gravitational waves during the merger of neutron star binaries



- computed in small tidal deformation approximation
- defined in buffer zone $\mathscr{R} \gg R \gg \mathscr{R}_*$ with $\mathscr{R}$ being the radius of curvature of the source of the perturbation

The I-Love-Q Relations by Yagi and Yunes





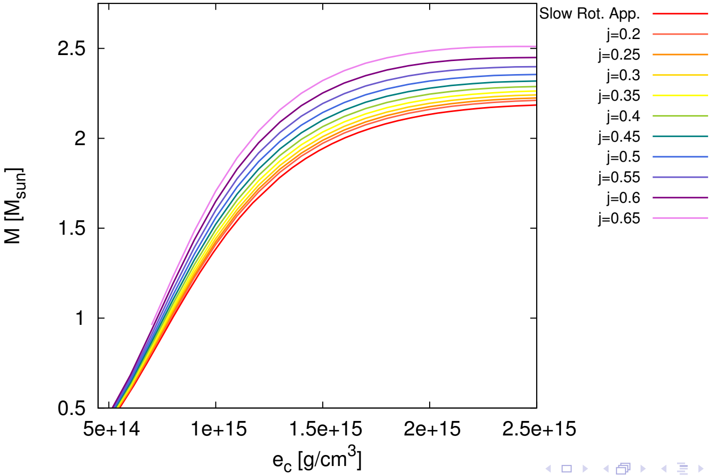
- Reduced Moment of inertia $\bar{I}$ versus the Kerr factor QM/J^2
- Reduced Moment of inertia $\bar{I}$ versus the tidal Love number $\bar{\lambda}^{tid}$

Fitting function

$$\ln y_i = a_i + b_i \ln x_i + c_i (\ln x_i)^2 + d_i (\ln x_i)^3 + e_i (\ln x_i)^4$$

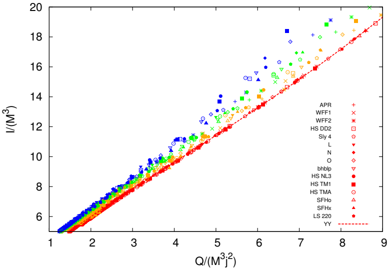
Rapid Rotation

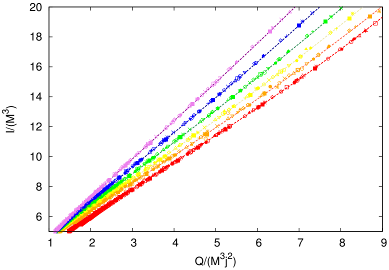
- equilibrium between gravitational, pressure and centrifugal forces
- Kepler angular velocity



Breakdown of Universal Relations

- Breakdown of universal relations



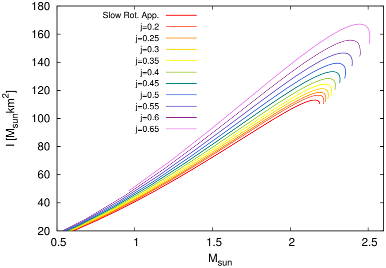


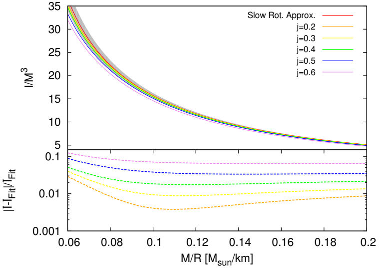
- $\bar{I} - \bar{Q}$-relation as a function of the observationally important (but dimensionful) rotational angular velocity

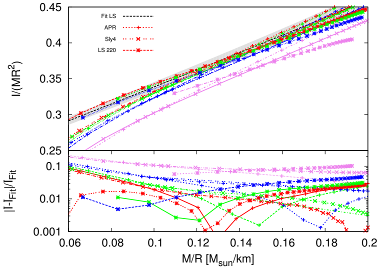
- rotation characterized by dimensionless spin parameter $j = J/M^2$

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Rapidly Rotating Models





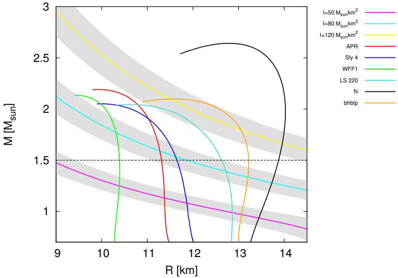


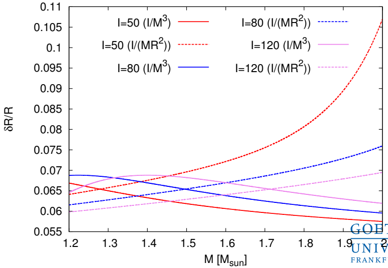
- dimensionless angular momentum $j = J/M^2$ instead of J

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Radius Measurement

- uncertainties in the modeling of atmospheres and radiation processes
- simultaneous measurement of mass and moment of inertia of a radio binary pulsar (e.g. PSR J0737-3039)
- determination of I up to 10 % accuracy through periastron advance and geodetic precession
- constraints on radius and EOS





GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Outlook

- newly born NS $\rightarrow$ fast differential rotation, high temperature
- Underlying physics?
- approach of limiting values of Kerr-metric
- dependence on internal structure far from the core $\rightarrow$ realistic EOSs are similar to each other
- approximation by elliptical isodensity contours
- modern EOSs are stiff $\rightarrow$ limit of an incompressible fluid

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Thank you!