Short-wave vortex instabilities in stratified flow

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Stratified Flow

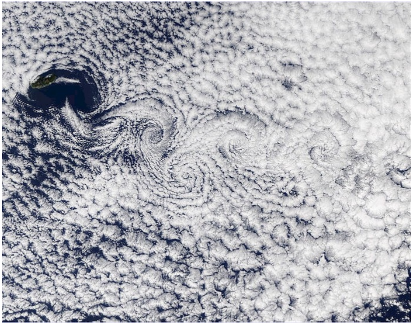


Figure : Image from NASA

Equations of Motion

Navier-Stokes for a compressible Newtonian fluid

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p - \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}, \tag{1}$$

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0. \tag{2}$$

$$\rho c_p \frac{DT}{Dt} - \alpha T \frac{D\rho}{Dt} = k \nabla^2 T + \phi, \tag{3}$$

where $\alpha = -\rho^{-1}(\partial\rho/\partial T)_p$ is the coefficient of thermal expansion, c_p is the specific heat, and ϕ represent conversion of kinetic energy to internal energy by viscous dissipation.
Equation of state,

$$\rho = \rho_0(1 - \alpha(T - T_0)). \tag{4}$$

Equations of motion

Incompressibility condition follows from:

- unsteadiness $\sim T^2 \gg L_v^2/v^2$,
- speed $\sim u^2/v^2 \ll 1$,
- gravity $\sim L_v \ll v^2/g$,

where T is characteristic time, L_v is characteristic vertical length scale, u is the characteristic velocity, v is the speed of sound in medium, g is gravitational constant.

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0 \Rightarrow \nabla \cdot \mathbf{u} = 0. \tag{5}$$

Equations of motion

Assume simple density form called the Boussinesq approximation

$$\rho(\mathbf{x}, t) = \rho_0 + \bar{\rho}(z) + \rho'(\mathbf{x}, t), \tag{6}$$

with $|\rho'| \ll |\bar{\rho}(z)| \ll |\rho_0|$.

$$\rho c_p \frac{DT}{Dt} - \alpha T \frac{D\rho}{Dt} = k\nabla^2 T + \phi, \tag{7}$$

becomes (after some thermodynamic approximations)

$$\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho' = \kappa \nabla^2 \rho' - \frac{\partial \bar{\rho}}{\partial z} w. \tag{8}$$

Equations of Motion

Boussinesq Equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0} \nabla p - \frac{\rho' g}{\rho_0} \hat{\mathbf{e}}_z + \nu \nabla^2 \mathbf{u}, \tag{9}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{10}$$

$$\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho' = \kappa \nabla^2 \rho' - \frac{\partial \bar{\rho}}{\partial z} w. \tag{11}$$

Define the buoyancy frequency or the Brunt-Väisälä frequency

$$N^2 = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}. \tag{12}$$

Equations of Motion

Non-dimensionalisation

$$\frac{D\mathbf{u}}{Dt} = -\nabla p - \rho' \hat{\mathbf{e}}_z + \frac{1}{Re}\nabla^2 \mathbf{u}, \tag{13}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{14}$$

$$\frac{D\rho'}{Dt} - \frac{w}{F_h^2} = \frac{1}{ReSc}\nabla^2 \rho', \tag{15}$$

Characteristic velocity U, length R, time-scale R/U, pressure $\rho_0 U^2$, density $\rho_0 U^2/gR$, and $Sc = \nu/\kappa$ the Schmidt number, ρ_0 is the background density, and g is the gravitational constant.

$$Re = \frac{UR}{\nu}, F_h = \frac{U}{NR}. \tag{16}$$

Respectively the Reynolds number and the Froude number.

Stability theory

- Ultimately interested in stratified turbulence but difficult.
- Initially study linear stability of flow, i.e. $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$ where $\mathbf{u}_0$ is a basic state and $|\mathbf{u}'| \ll |\mathbf{u}_0|$.
- Linear stability can give insight into important mechanisms.
- Mechanisms can be probed further using nonlinear simulations.

Zigzag Instability

- Billant and Chomaz (2000) discovered new instability unique to stratified flow.
- Confirmed experimentally, theoretically, numerically.
- Named “zigzag” instability due to structure.
- Emerges at buoyancy scale U/N where U is velocity, N is Brunt-Väisälä frequency.

Zigzag instability

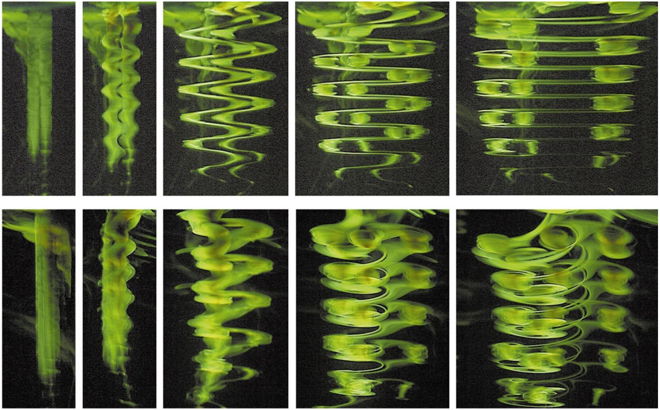


Figure : From Billant and Chomaz (2000a). From left to right the pictures are taken at $7, 36, 75, 109, 121, 176$ seconds after the flaps have closed. Top is frontal view, Bottom is side view.

Our work

- Further analysis (e.g. Deloncle et al. 2011, Waite 2012) has shown the importance of the buoyancy length scale U/N.
- Sub-buoyancy scale remains unexplored.
- Nature excites scales well below the buoyancy scale.
- Investigate linear and nonlinear evolution of sub-buoyancy scales.

Numerical schemes

- Spectral method with 2/3-rule dealiasing.
- Adams-Bashforth 2nd and 3rd order time-stepping.
- Diffusion term integrated exactly.
- Hyperviscosity.
- Initial state a Lamb-Chaplygin dipole subject to random noise.

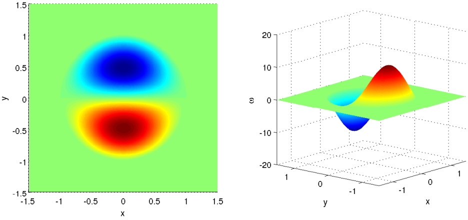


Figure : Lamb-Chaplygin Dipole.

Numerical schemes

From Fourier analysis

$$\frac{d^n f}{dx^n} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (ik)^n \hat{f}(k) e^{ikx} dk \tag{17}$$

Simple algorithm to compute derivatives

1. Compute $\hat{f}(k)$ from $f(x)$
2. Multiply $\hat{f}(k)$ by $(ik)^n$
3. Invert $(ik)^n \hat{f}(k)$ to obtain $f^{(n)}(x)$

Compute Fourier transforms using FFTs in $\mathcal{O}(n \log n)$.

Numerical schemes

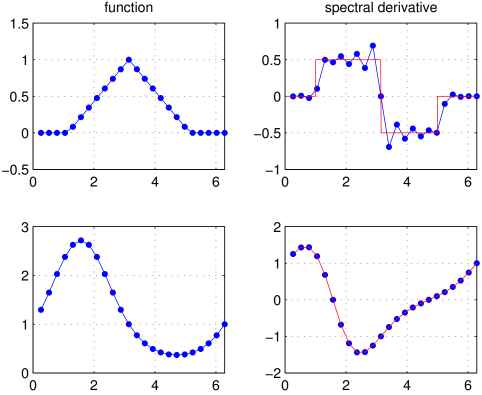


Figure : Adapted from Trefethen. $n = 24$ grid points used. Red curve represents the exact derivative.

Dealiasing

Need to compute terms like

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}. \tag{18}$$

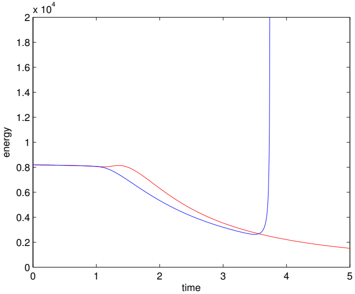
Potential algorithm

1. Transform the Fourier coefficients to real space
2. Multiply terms grid wise
3. Transform back to Fourier space

Simple, but has problems due to aliasing. Can be fixed by removing 1/3 of the coefficients.

Dealiasing

Solution to the viscous Burgers equation using spectral (red) and pseudospectral (blue).



$$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} = \nu \frac{\partial^2 \psi}{\partial x^2}. \tag{19}$$

Diffusion Term

Navier-Stokes in Fourier space

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + \mathcal{F}(\mathbf{u} \cdot \nabla \mathbf{u}) = -\frac{1}{\rho_0}\mathbf{k}\hat{p} - \nu k^2 \hat{\mathbf{u}}, \mathbf{k} \cdot \hat{\mathbf{u}} = 0. \tag{20}$$

Take the dot product with $\mathbf{k}$ and using the orthogonality condition we obtain

$$\mathbf{k} \cdot \mathcal{F}(\mathbf{u} \cdot \nabla \mathbf{u}) + \frac{1}{\rho_0} k^2 \hat{p} = 0. \tag{21}$$

Isolating for pressure and substituting back in

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + \mathcal{F}(\mathbf{u} \cdot \nabla \mathbf{u})(\mathbf{1} - \frac{\mathbf{k}\mathbf{k}}{k^2}) = -\nu k^2 \hat{\mathbf{u}}. \tag{22}$$

Diffusion Term

Equation is of the form.

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + \nu k^2 \hat{\mathbf{u}} = F(\hat{\mathbf{u}}), \tag{23}$$

and can be rewritten as

$$\frac{\partial}{\partial t}(\hat{\mathbf{u}} e^{\nu k^2 t}) = e^{\nu k^2 t} F(\hat{\mathbf{u}}). \tag{24}$$

Thus the diffusion term is exact.

Hyperviscosity

Simulating high Reynolds number flow is difficult. Replace diffusion term with higher order

$$\nu k_{max}^2 = \nu_i k_{max}^i, \tag{25}$$

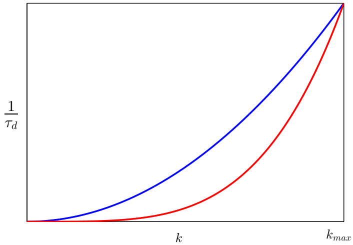


Figure : The inverse diffusion times, $1/\tau_d$, of the wavenumbers for the regular viscosity, blue, and the hyperviscosity case, red.

Growth Rates

Leading eigenmodes grow as

$$\mathbf{u}, \rho \propto C(x, y)e^{\sigma t}, \tag{26}$$

and we can obtain the largest growth rate by the formula

$$\sigma = \lim_{t \to \infty} \frac{1}{2} \frac{d \ln E}{dt}, \tag{27}$$

$E \propto u^2 + v^2 + w^2$.
Alternative: Krylov methods.

Linear results

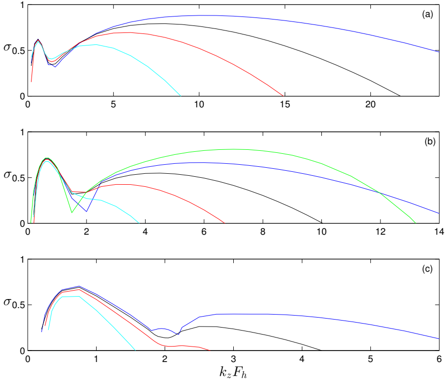


Figure : Growth rate σ for fixed F_h =(a) 0.2, (b) 0.1, (c) 0.05 with Re= 2000 (cyan), Re= 5000 (red), Re= 10,000 (black), Re= 20,000 (blue). In panel (b) green line is hyperviscosity with $Re = 20{,}000$.

Linear results

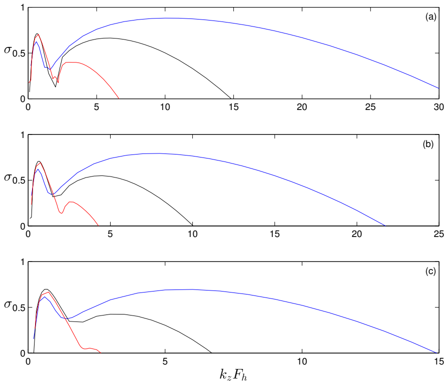


Figure : Growth rate σ for fixed Re $= (a)20{,}000, (b)10{,}000, (c)5000$ with $F_h = 0.05$ (red), $F_h = 0.1$ (black), $F_h = 0.2$ (blue).

Linear results

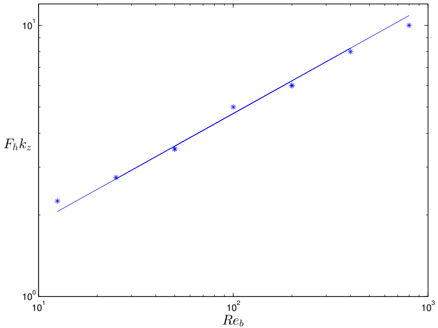


Figure : The location of the second peak as a function of the buoyancy Reynolds number $Re_b = ReF_h^2$. The straight line is $Re_b^{2/5}$.

Linear results

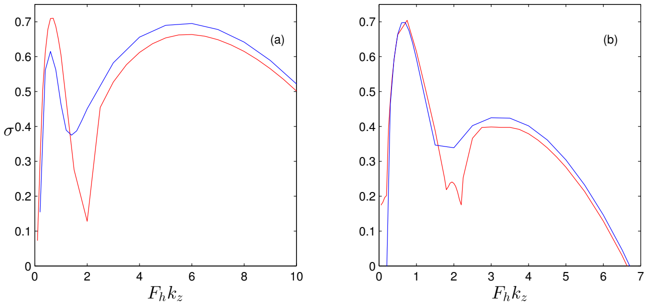


Figure : Growth rate σ for fixed Re_b. In (a), red is $Re = 20{,}000, F_h = 0.1$ and blue is $Re = 5000, F_h = 0.2$, both corresponding to $Re_b = 500$; in (b) red is $Re = 20{,}000, F_h = 0.05$ and blue is $Re = 5000, F_h = 0.1$, both corresponding to $Re_b = 50$.

Linear results

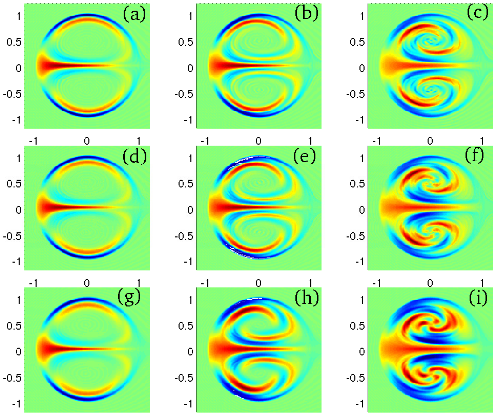


Figure : Perturbation vertical vorticity ω_z at second peak for $Re = 20{,}000$ (top) , 10.000 (middle) , 5000 (bottom) ; and $F_h = 0.2$ (left) , 0.1 (middle) , 0.05 (right) .

Linear results

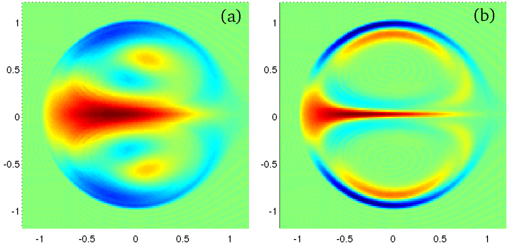


Figure : Perturbed vertical vorticity ω_z at (a) the zigzag peak (b) the second peak for $Re = 5000, F_h = 0.2$.

Nonlinear results

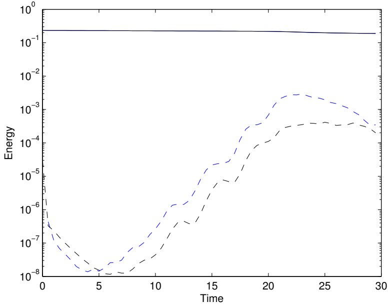


Figure : Time series demonstrating the two ways of computing the energy for $Re = 5000, F_h = 0.2$, and $k_z = 40$. The blue curves correspond to the kinetic energy separated into 2D (solid) and 3D (dashed); the black curves are the total kinetic energy (solid) and potential energy (dashed). All energies are domain averages.

Nonlinear results

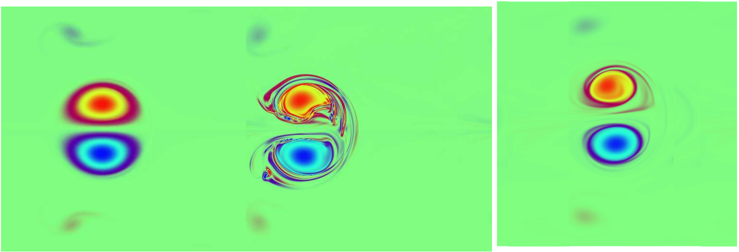


Figure : Evolution of the vertical vorticity for $Re = 5000, F_h = 0.2, k_z = 40$ for $t = 15$ (top right), $t = 20$ (top left), $t = 25$ (bottom). Red corresponds to maximum vorticity and blue corresponds to minimum vorticity.

Nonlinear results

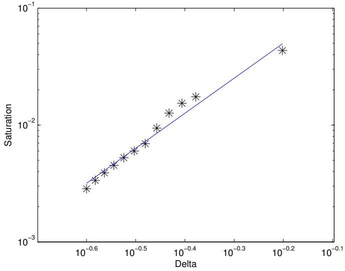


Figure : Saturation levels for a range of aspect ratios δ for $Re = 2000$ and $F_h = 0.2$. The curve has slope 3.

$\delta = L_v/L_h$ is the aspect ratio. Saturation is E_{2D}/E_{3D}.

Conclusions

- Linear simulations predict sub-buoyancy scale instability.
- Short-wave instability exhibits growth rates similar to zigzag.
- Nonlinear simulation suggest saturation as δ^3.

Future Investigations

- Examine sub-buoyancy scales in other models.
- Investigate wakes behind dipole.
- Sub-dominant modes.
- Oscillatory regime.