Pinhole Camera Visualisations of Accretion Disks around Kerr Black Holes

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Contents

The Einstein equations

The metric tensor is determined by the Einstein equations:

$$G^{\mu}{}_{\nu} + g^{\mu}{}_{\nu}\Lambda = 8\pi T^{\mu}{}_{\nu} \ . \tag{1}$$

In vacuum on small scales they reduce to

$$R^{\mu}{}_{\nu} = 0 \ . \tag{2}$$

Schwarzschild and Kerr spacetime

Axisymmetric stationary Kerr spacetime:

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma^2}\right)\mathrm{d}t^2 - \frac{4Mar}{\Sigma^2}\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\phi + \frac{\Sigma^2}{\Delta}\mathrm{d}r^2 + \Sigma^2\,\mathrm{d}\theta^2 + \frac{A}{\Sigma^2}\sin^2\theta\,\mathrm{d}\phi^2 \;, \tag{3}$$

where

$$\Sigma^2 = r^2 + a^2\cos^2\theta \;, \tag{4}$$

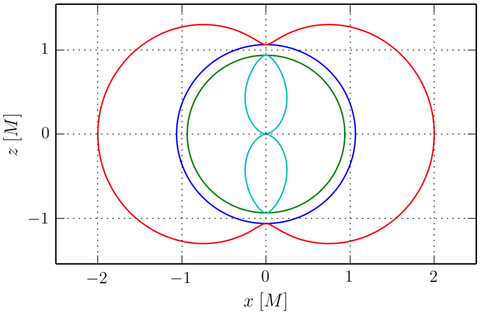
$$\Delta = r^2 - 2Mr + a^2 \;, \tag{5}$$

$$A = \left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta \;. \tag{6}$$

Horizons

[Bardeen 1972, Visser 2008, Younsi 2013]:

- red: static limit (r_0)
- dark-blue: outer event horizon (r_+)
- green: inner event horizon (r_-)
- turquoise: inner static limit ($\tilde{r}_-$)
- only physical singularity at: $\Sigma^2 := r^2 + a^2 \cos^2\theta = 0$.



Geodesic equations in Boyer-Lindquist coordinates

Using $2L = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ (Lagrangian of a freely falling test particle) one obtains the following equations for geodesics in Kerr spacetime [Younsi 2013]:

$$\dot{t} = E + \frac{2Mr}{\Sigma^2\Delta}\left[\left(r^2 + a^2\right)E - al_z\right] , \tag{7}$$

$$\dot{r}^2 = \frac{\Delta}{\Sigma^2}\left(\mu + E\dot{t} - l_z\dot{\phi} - \Sigma^2\dot{\theta}^2\right) , \tag{8}$$

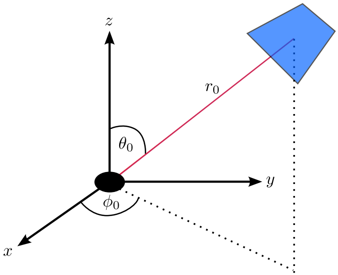
$$\dot{\theta}^2 = \frac{1}{\Sigma^4}\left[Q + \left(E^2 + \mu\right)a^2\cos^2\theta - l_z^2\cot^2\theta\right] , \tag{9}$$

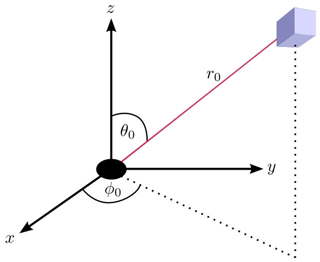
$$\dot{\phi} = \frac{2aMrE + \left(\Sigma^2 - 2Mr\right)l_z\csc^2\theta}{\Sigma^2\Delta} . \tag{10}$$

Geodesic equations in Boyer-Lindquist coordinates

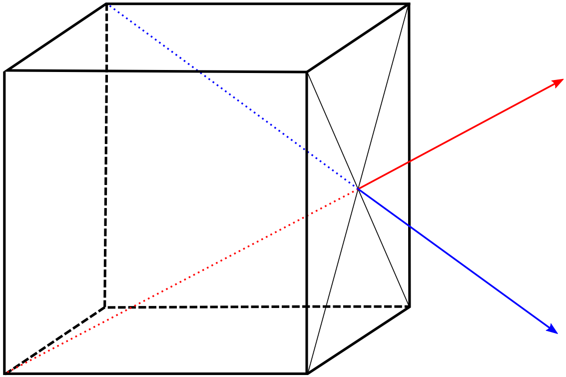
- Move to second derivatives of r and θ to avoid the squares.
  $\Rightarrow$ 6 coupled differential equations $(\dot{t}, \dot{r}, \ddot{r}, \dot{\phi}, \dot{\theta}, \ddot{\theta})$.
- Solved by fourth order Runge-Kutta with fifth order Runge-Kutta error precision control.

Image-plane method vs. pinhole camera method





Pinhole camera method



What are tetrads?

A tetrad is a set of orthonormal basis vectors [Misner et al. 1973]

$$\underline{e}_{(\hat{\mu})} \cdot \underline{e}_{(\hat{\nu})} = \eta_{\hat{\mu}\hat{\nu}} \ . \tag{11}$$

Counter-example: Basis vectors in Boyer-Lindquist coordinates are neither normalised, nor orthogonal to each other:

$$\underline{e}_{(\mu)} \cdot \underline{e}_{(\nu)} = g_{\alpha\beta} \underline{e}^{\alpha}_{(\mu)} \underline{e}^{\beta}_{(\nu)} = g_{\alpha\beta} \delta_{\mu}{}^{\alpha} \delta_{\nu}{}^{\beta} = g_{\mu\nu} \ . \tag{12}$$

Constructing a stationary observer's tetrad

Define the pinhole camera's four-velocity as

$$\underline{e}_{(\hat{t})} := (u^\mu) = \begin{pmatrix} u^t(x^\mu) \\ \vec{o} \end{pmatrix} = e^t_{(\hat{t})} \underline{e}_{(t)} \ , \tag{13}$$

where it was used that $e^t_{(\hat{t})} = u^t$ [Misner et al. 1973]. This yields:

$$-1 \stackrel{!}{=} \underline{e}_{(\hat{t})} \cdot \underline{e}_{(\hat{t})} = \left(e^t_{(\hat{t})} \right)^2 \underline{e}_{(t)} \underline{e}_{(t)} \tag{14}$$

$$\Rightarrow e^t_{(\hat{t})} = \frac{1}{\sqrt{-g_{tt}}} e^t_{(t)} \tag{15}$$

Constructing a stationary observer's tetrad

Similarly suitable $\hat{\theta}$ and $\hat{r}$ vectors are obtained:

$$\underline{e}_{(\hat{r})} = \frac{1}{\sqrt{g_{rr}}}\underline{e}_{(r)} = \frac{\sqrt{\Delta}}{\Sigma}\underline{e}_{(r)} \ , \tag{16}$$

$$\underline{e}_{(\hat{\theta})} = \frac{1}{\sqrt{g_{\theta\theta}}}\underline{e}_{(\theta)} = \frac{1}{\Sigma}\underline{e}_{(\theta)} \ . \tag{17}$$

The ansatz for the ϕ-basis vector must respect the off-diagonal terms of the Kerr metric in BL coordinates:

$$\underline{e}_{(\hat{\phi})} = K\left(x^i\right)\underline{e}_{(\phi)} + P\left(x^i\right)\underline{e}_{(t)} \ . \tag{18}$$

Constructing a stationary observer's tetrad

The orthonormality conditions give two equations, which are used to determine K and P. This finally yields:

$$\underline{e}_{(\hat{\phi})} = \frac{\sqrt{-g_{tt}}}{\sqrt{\Delta}\sin\theta} \left(\underline{e}_{(\phi)} - \frac{g_{\phi t}}{g_{tt}} \underline{e}_{(t)} \right) . \tag{19}$$

Initialising the four-momenta

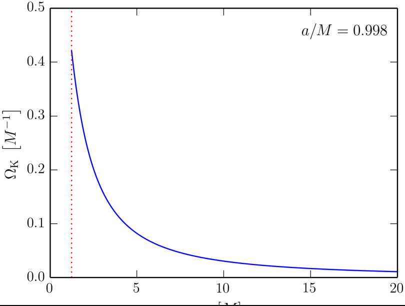
Pinhole camera method by Bohn et al. (2014):

$$\xi^{\mu} = C \; e^{\mu}_{(\hat{t})} - e^{\mu}_{(\hat{r})} - \frac{\beta}{r_0} \; e^{\mu}_{(\hat{\theta})} - \frac{\alpha}{r_0} \; e^{\mu}_{(\hat{\phi})} \tag{20}$$

- C normalisation constant, which is determined by $\underline{p} \cdot \underline{p} = \mu$ ($\mu = 0$ for photons and $\mu = -1$ for massive particles).
- r_0 causes that the pinhole camera zooms in on the accretion disk.
- α, β are spaced equidistant in the interval $[-22M, 22M]$.

Angular velocity profile

- assume fast rotating disk, trajectories of emitters approximated by circular motion [cf. Shakura and Sunyaev 1973]
- inner edge of disk at ISCO: $r_{\mathrm{inner}} = r_{\mathrm{ISCO}}$; outer edge: $r_{\mathrm{outer}} = 20M$.



Particles on circular orbits in the equatorial plane have an angular velocity of [Bardeen et al. 1972]

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \Omega_{\mathrm{K}} = \frac{\sqrt{M}}{a\sqrt{M} + r^{3/2}} \ . \quad (21)$$

Redshift

Define fluid's and observer's four-velocity:

$$\begin{aligned}\left(u^{\mu}_{\mathrm{fluid}}\right) &= (u^{t}_{\mathrm{fluid}}, 0, 0, u^{\phi}_{\mathrm{fluid}}) \\ &= u^{t}_{\mathrm{fluid}}(1, 0, 0, \Omega_K)\ , \end{aligned} \tag{22}$$

$$\left(u^{\mu}_{\mathrm{obs}}\right) = (u^{t}_{\mathrm{obs}}, 0, 0, 0)\ . \tag{23}$$

Using $E_{\mathrm{fluid}} = -p_\alpha u^{\alpha}_{\mathrm{fluid}}$ one obtains:

$$g = 1 + z := \frac{E_{\mathrm{fluid}}}{E_{\mathrm{obs}}} = \frac{\dot{t}_{\mathrm{fluid}}}{\dot{t}_{\mathrm{obs}}}\left(1 - \Omega_{\mathrm{K}}\frac{l_z}{E}\right) \tag{24}$$

[Courtesy of Dr. Younsi, who kindly gave me this derivation].

General relativistic radiative transfer (GRRT) equations

- Computation of intensity by solution of the GRRT equations [Fuerst and Wu 2004, Baschek et al. 1997]:

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\lambda} = -p_\alpha u^\alpha|_\lambda \left(-\alpha_{0,\nu}\mathcal{I} + \frac{j_{0,\nu}}{\nu_0^3} \right) \ . \tag{25}$$

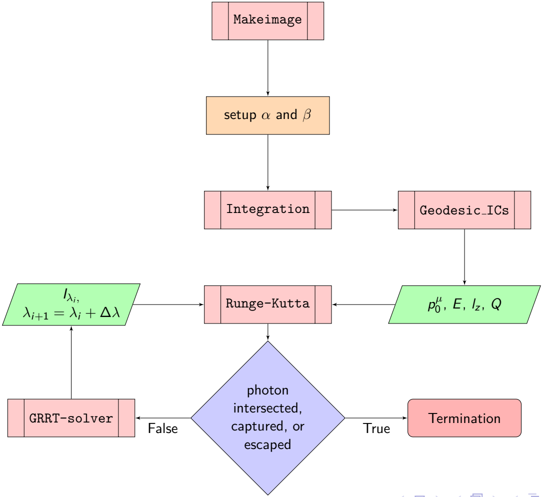
Equivalent formulation [Younsi 2013]:

$$\frac{\mathrm{d}\tau_\nu}{\mathrm{d}\lambda} = g\alpha_{0,\nu} \ , \tag{26}$$

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\lambda} = g \left(\frac{j_{0,\nu}}{\nu_0^3} \right) \mathrm{e}^{-\tau_\nu} \ . \tag{27}$$

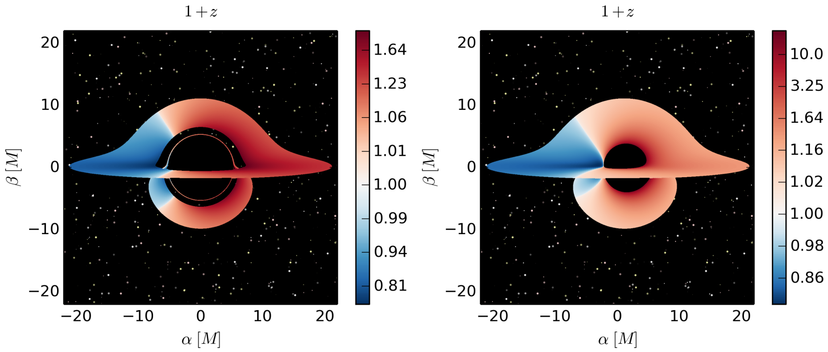
- Lorentz-invariant intensity [Rybicki and Lightman 2004, Younsi 2013]: $\mathcal{I} = I_\nu/\nu^3$; specific intensity: I_ν
- optical depth [Baschek et al. 1997, Younsi 2013]:
  $\tau_\nu(\lambda) = -\int\limits_{\lambda_0}^{\lambda} \alpha_{0,\nu}(\lambda') p_\alpha u^\alpha|_{\lambda'} \, \mathrm{d}\lambda'$,
  where α_ν is the absorption, defined by [Rybicki and Lightman 2004] $\mathrm{d}I_\nu = -\alpha_\nu I_\nu \, \mathrm{d}s$, with the distance $\mathrm{d}s$ the beam travels
- assumed vanishing absorption outside the disk: $\alpha_{0,\nu} \overset{!}{=} 0$
- ansatz for the emissivity coefficient: $j_{0,\nu} \propto r^{-3}$.

Code structure

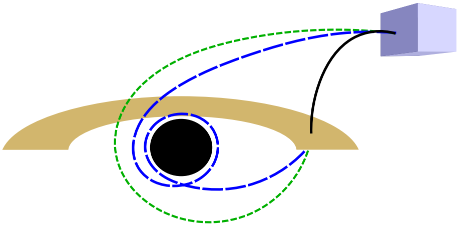


Redshift – $(r, \theta) = (1000\ M, 85°)$

- left: $a/M = 0$, right: $a/M = 0.998$



Higher order photons

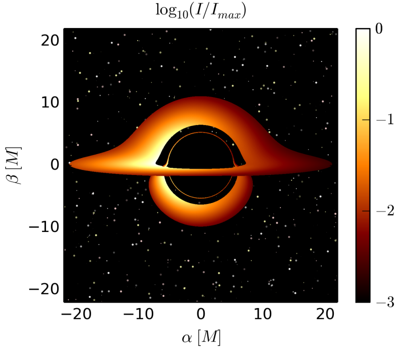


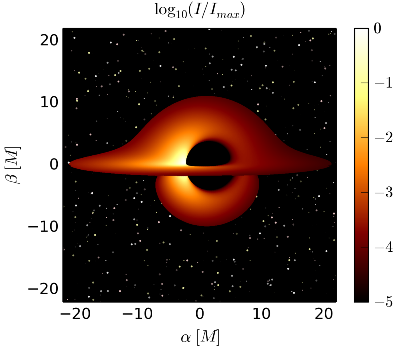
Higher order photons can make several turns around the black hole.

This yields the characteristic photon rings.

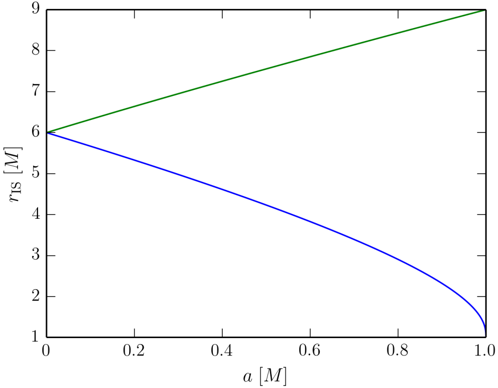
Intensity – $(r, \theta) = (1000\ M, 85^\circ)$

- left: $a/M = 0$, right: $a/M = 0.998$





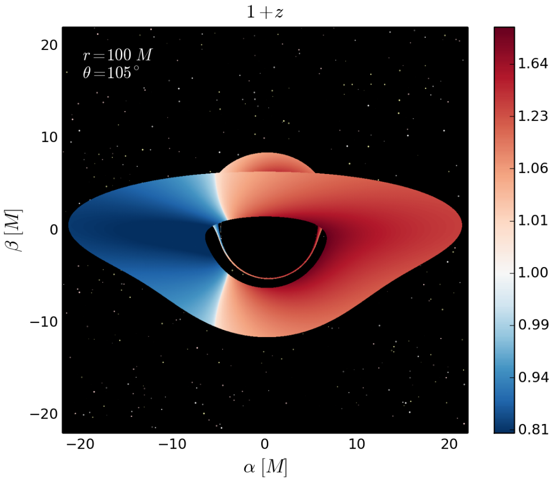
Inner most stable circular orbit (ISCO) [Bardeen 1972]



Green line for counter-
blue line for co-rotating
(massive) particles.

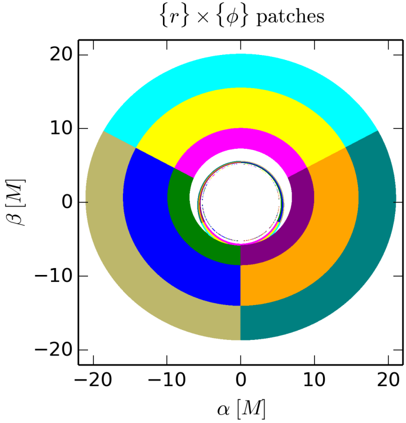
In the Schwarzschild limit the ISCO for co-rotating particles coincides with the ISCO of counter-rotating ones.

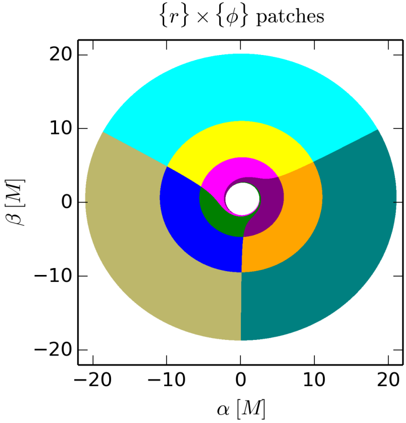
Rotation in θ – animation, $a/M = 0.5$



Coordinate patches – $(r, \theta) = (1000M, 25^\circ)$

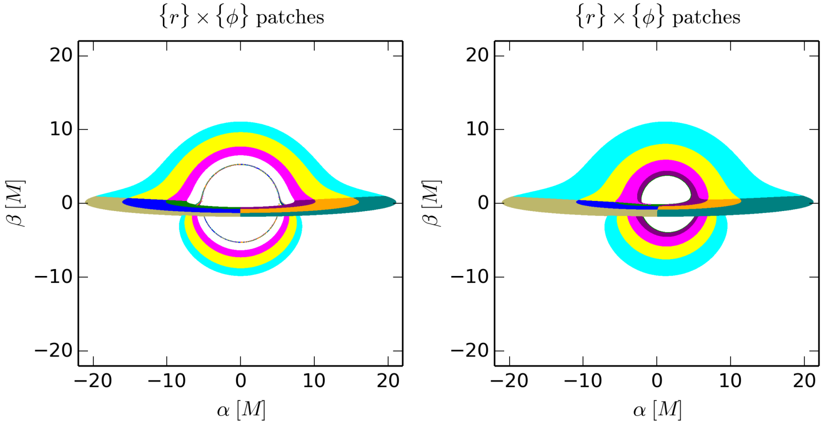
- left: $a/M = 0$, right: $a/M = 0.998$





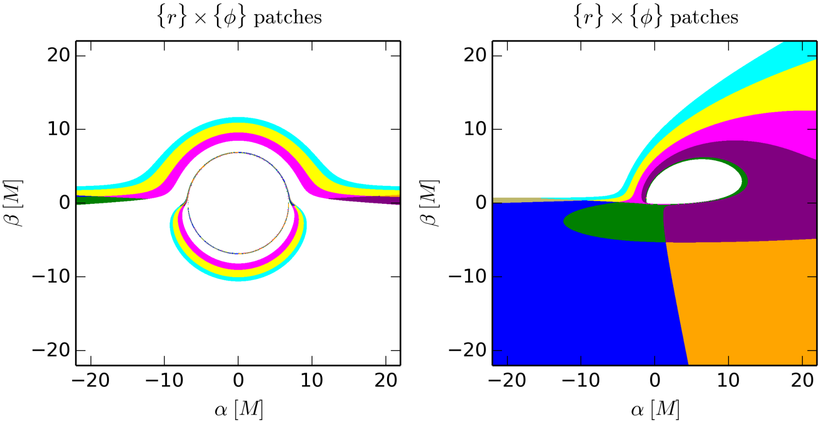
Coordinate patches – $(r, \theta) = (1000M, 85^\circ)$

- left: $a/M = 0$, right: $a/M = 0.998$

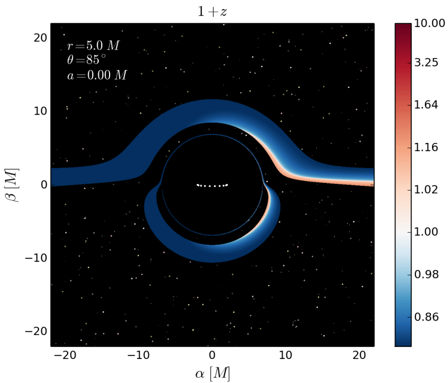


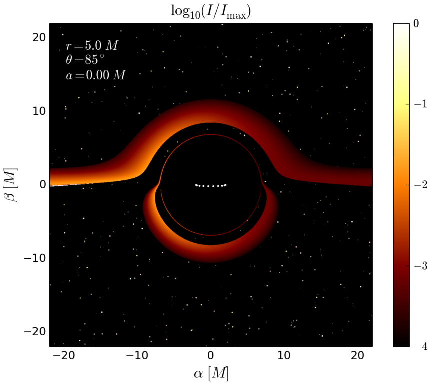
Coordinate patches – $(r, \theta) = (5M, 85^\circ)$

- left: $a/M = 0$, right: $a/M = 0.998$



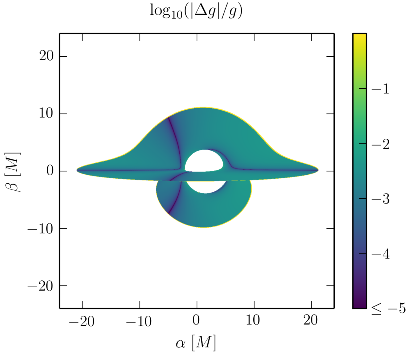
Up-spinning black hole

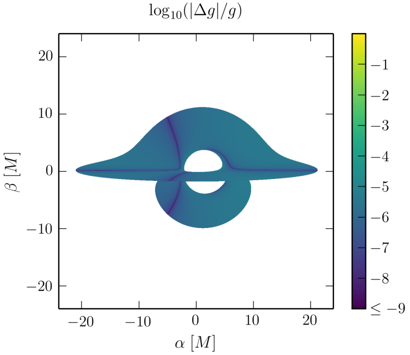




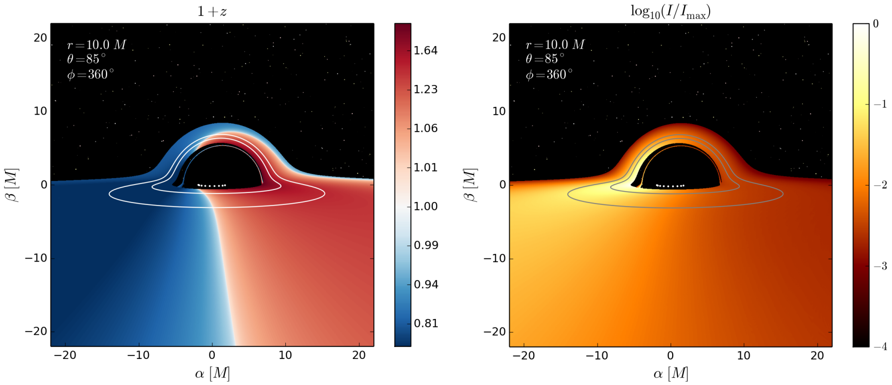
Comparison of $1 + z$ – left: $r = 10^3 M$, right: $r = 10^6 M$

- $a/M = 0.998$, $\theta = 85°$





Hyperbolic Orbit – $a/M = 0.5$



Conclusion

- created a method that is able to visualise the radiation emanating from an accretion flow
- visualisation tool for curved spacetimes, basically applicable to any other spacetime
- provides visual description of the GR effects happening close to the horizon (using proper coordinates even below)

Conclusion – future work

- using other disk models/accretion flows
- implement gravitational lensing of background
- reconstruction of pinhole camera with comoving tetrad
- reformulation of code in 3+1 decomposition of spacetime
- considering other spacetimes, e.g. binary black holes.

Thank you for your attention!

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