Modelling the final spin from black hole mergers

Bachelor Thesis

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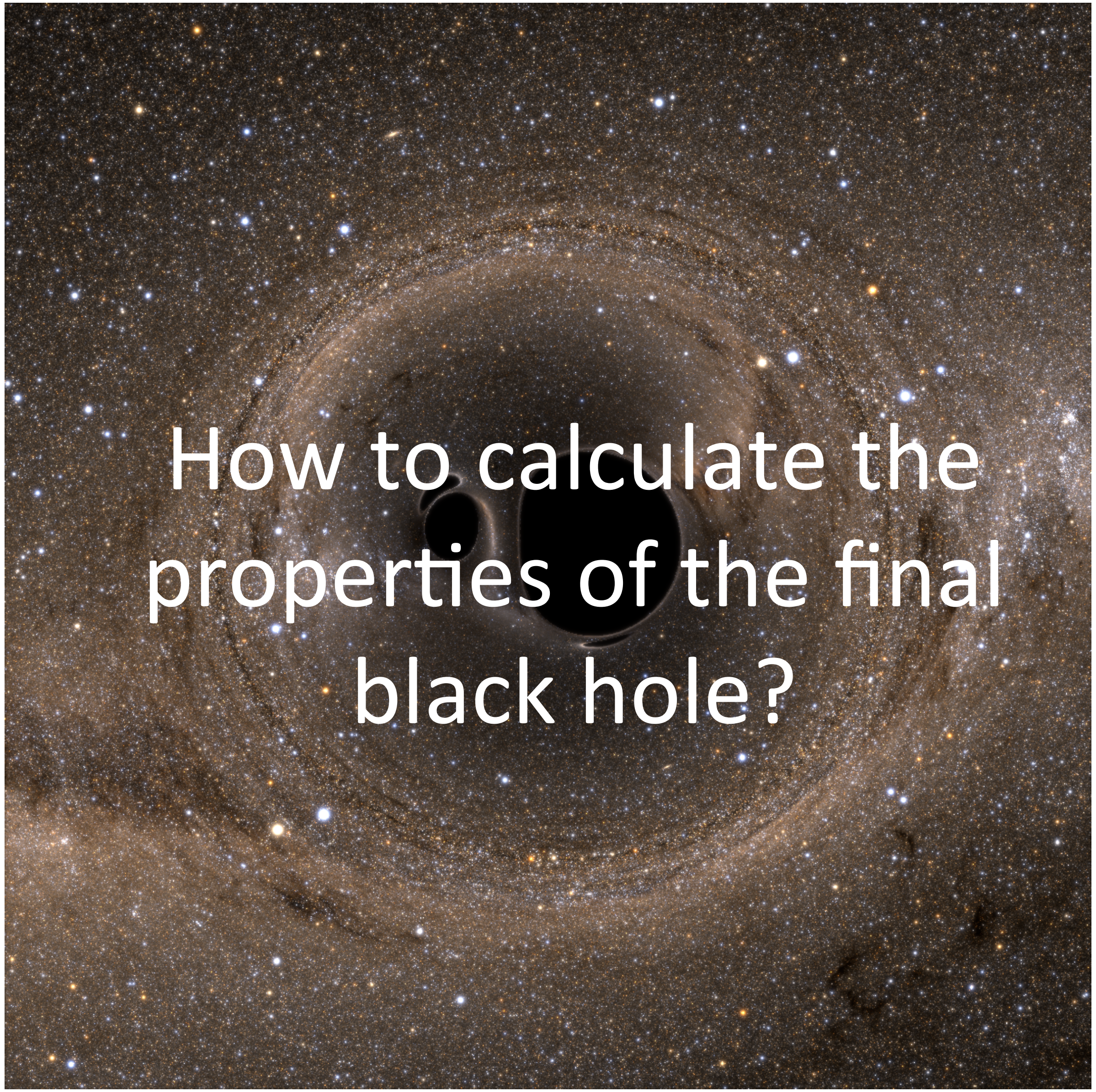

How to calculate the properties of the final black hole?

Initial data

- Mass ratio $q = M_2/M_1$
  or symmetric mass ratio
  $v = q/(1+q)^2$
- Initial spin $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$
  with $\boldsymbol{a}_i = \boldsymbol{J}_i / M_i^2$

} Seven dimensional parameter space

Aim:

Derive a phenomenological formula such as $\boldsymbol{a}_{fin} = \boldsymbol{a}_{fin}(\boldsymbol{a}_1, \boldsymbol{a}_2, v)$

AEI Formula

Deriving a formula for black hole binaries with aligned and equal spins $(a_1 = a_2 = a)$ using polynomial expansion:

$$a_{fin} = s_0 + s_1 a + s_2 a^2 + s_3 a^3 + s_4 a^2 v + s_5 a v^2 + t_0 a v + t_1 v + t_2 v^2 + t_3 v^3$$

Setting coefficient constraints to simplify

- $a_{fin}(a, \nu=0) = a$ → $s_0 = s_2 = s_3 = 0, \; s_1 = 1$
- Taylor expansion at $a_{fin}(a, \nu \ll 1)$ → $t_1 = \left|\frac{\partial a_{fin}}{\partial \nu}\right|_{(a=0,\ \nu=0)} = 2\sqrt{3}$
- accurate simulation for $a=0, \nu=0.25$

$$a_{fin} = \frac{\sqrt{3}}{2} + \frac{t_2}{16} + \frac{t_3}{64} = 0.68646 \pm 0.00004$$

→ $t_2 = 16 \cdot (0{,}68646 - \frac{t_3}{64} - \frac{\sqrt{3}}{2}) \pm 0{,}00064$

Expansion for unequal spins

$$a \rightarrow \tilde{a} \equiv (a_1 + a_2 q^2) / (1 + q^2)$$

AEI formula for aligned binaries

$$a_{fin} = \tilde{a} + \tilde{a}\nu(s_4\tilde{a} + s_5\nu + t_0) + \nu(2\sqrt{3} + t_2\nu + t_3\nu^2)$$

$s_4 = -0.1229$, $s_5 = 0.4537$,
$t_0 = -2.8904$, $t_3 = 2.5763$

The four unconstraint coefficients were fitted to 72 numerical simulations from papers published up to 2009.

Modelling generic binaries

Starting point: $$\boldsymbol{S}_{fin} = \boldsymbol{S}_1 + \boldsymbol{S}_2 + \tilde{\boldsymbol{l}}$$

5 assumptions for expanding the AEI formula for generic binaries

I. $M_{fin} = M_1 + M_2$

II. $\partial_r \left|\boldsymbol{S}_1\right| = \partial_r \left|\boldsymbol{S}_2\right| = \partial_r \left|\tilde{\boldsymbol{l}}\right| = 0$

III. $\boldsymbol{S}_{fin} \parallel \boldsymbol{J}(r_{In})$

IV. $\partial_r \left(\hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{L}}\right) = \partial_r \left(\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2\right) = 0$

(III and IV: Adiabatic approximation)

V. $a_{fin}(a_1 = -a_2, \nu) = a_{fin}(a_1 = a_2 = 0, \nu)$

Using the assumptions and

$$\boldsymbol{S}_{fin} = \boldsymbol{S}_1 + \boldsymbol{S}_2 + \tilde{\boldsymbol{l}}$$

one can derive the norm of the final spin

$$|\boldsymbol{a}_{fin}| = \frac{1}{(1+q)^2}\left(|\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2 q^4 + 2|\boldsymbol{a}_1||\boldsymbol{a}_2| q^2 \cos\alpha + 2\left(|\boldsymbol{a}_1|\cos\tilde{\beta}(r_{In}) + |\boldsymbol{a}_2| q^2 \cos\tilde{\gamma}(r_{In})\right)|\boldsymbol{l}| q + |\boldsymbol{l}|^2 q^2\right)^{1/2}$$

Calculated at the initial separation,

$\cos\tilde{\beta} = \hat{\boldsymbol{a}}_1 \cdot \hat{\boldsymbol{L}}$ and $\cos\tilde{\gamma} = \hat{\boldsymbol{a}}_2 \cdot \hat{\boldsymbol{L}}$

Remaining quantity

Note that this expression of the norm is independent of the radius.

Matching with the AEI formula for aligned spins leads to

$$|\boldsymbol{l}| = 2\sqrt{3} + t_2 v + t_3 v^2 + \frac{s_4}{(1+q^2)^2}(|\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2 q^4 + 2|\boldsymbol{a}_1||\boldsymbol{a}_2| q^2 \cos\alpha))$$
$$+(\frac{s_5 v + t_0 + 2}{1+q^2})(|\boldsymbol{a}_1|\cos\tilde{\beta}(r_{In}) + |\boldsymbol{a}_2| q^2 \cos\tilde{\gamma}(r_{In}))$$

The direction of the final spin is given by $\boldsymbol{S}_{fin} \| \boldsymbol{J}(r_{In})$

This formula was successfully proven and compared with the data published up to 2009.
Problem:

Size and *accuracy* of the fitting data set

Improvement of the AEI formula

- Gathering of 272 new BBH simulations with spin aligned (in addition to the 72 “old” simulations) → **fitting data**
- Gathering of 440 new generic BBH (in addition to the 86 “old” simulations) → **validation data**
- Introduction of the fourth order term in the initial polynomial expansion for equal, aligned spins

$$a_{fin} = s_0 + s_1 a + s_2 a^2 + s_3 a^3 + \underline{s_6 a^4} + s_4 a^2 v + s_5 a v^2 + \underline{s_7 a^3 v} + \underline{s_8 a^2 v^2} + \underline{s_9 a v^3} + \\ + t_0 a v + t_1 v + t_2 v^2 + t_3 v^3 + \underline{t_4 v^4}$$

  and derivation of the generic expression from there

- Sampling the fitting data set

Data Fit

Fitting the formula without additional terms (only AEI terms) → GU formula

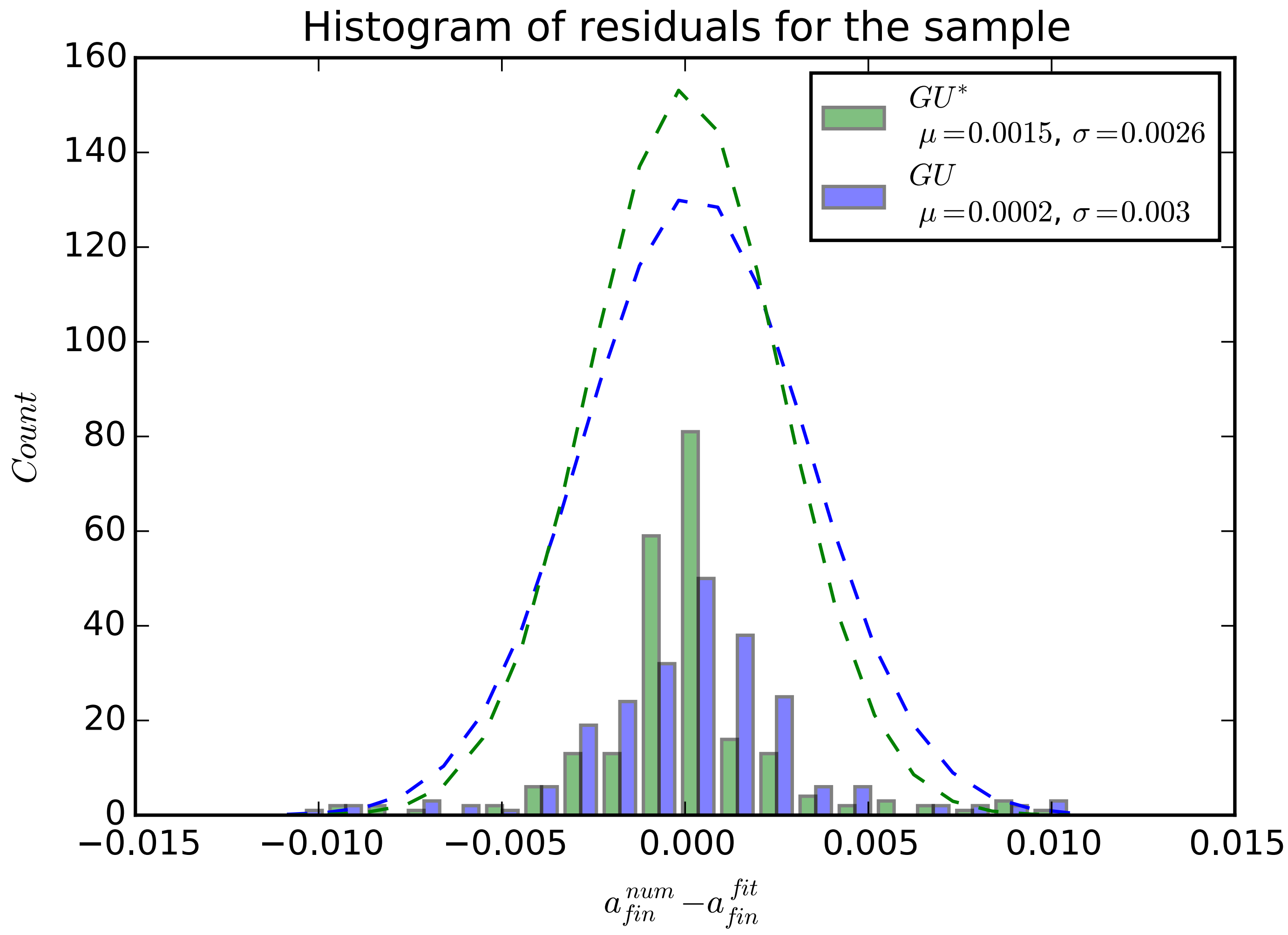
$$a_{fin} = \tilde{a} + \tilde{a}v(s_4\tilde{a} + s_5 v + t_0) + v(2\sqrt{3} + t_2 v + t_3 v^2)$$

Fitting the formula with additional terms → GU* formula

$$a_{fin} = \tilde{a} + \tilde{a}v(s_4\tilde{a} + s_5 v + s_7\tilde{a}^2 + s_8\tilde{a}v + s_9 v^2 + t_0) + 2\sqrt{3}v + t_2 v^2 + t_3 v^3 + t_4 v^4$$

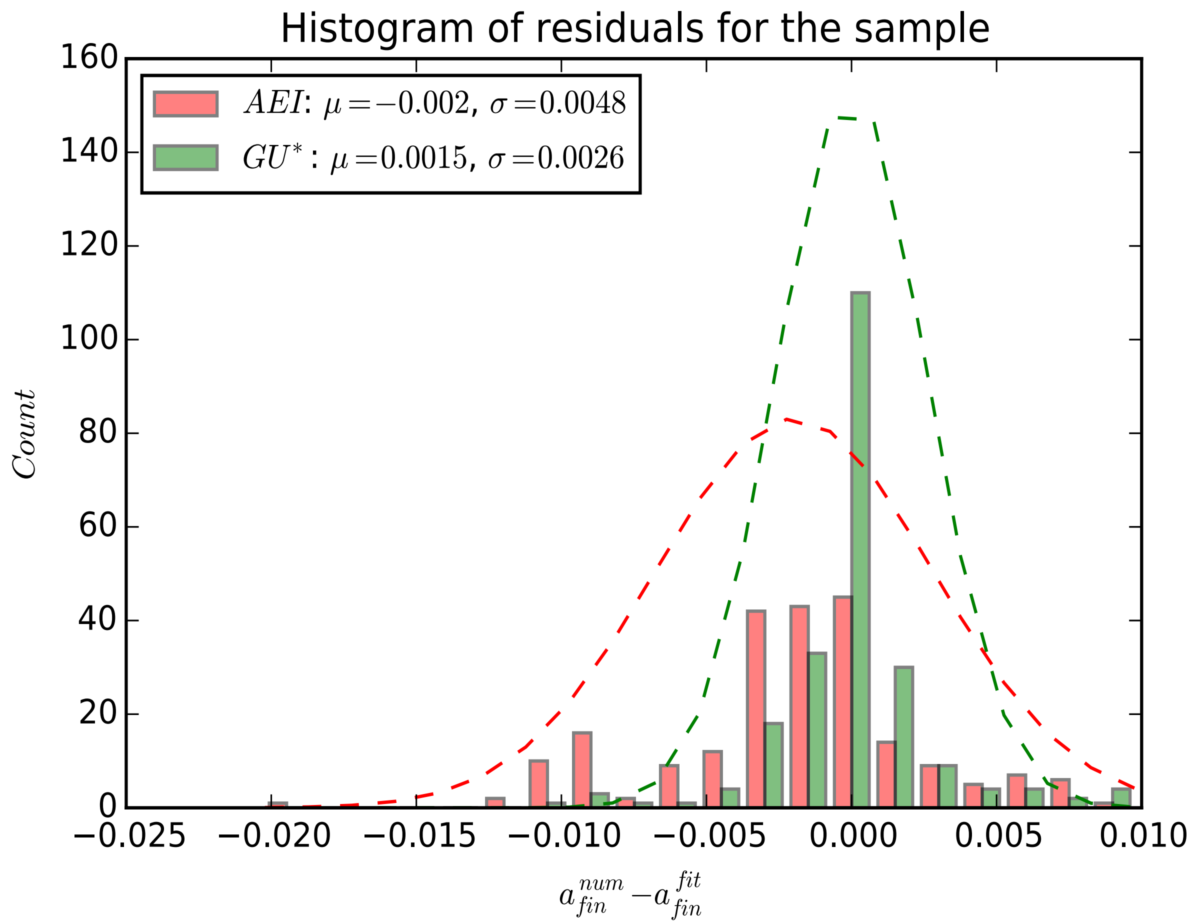
From the fitting data set with aligned spin, sample such that only *recent* simulations (since 2010) are included and simulations *with too large residual are not considered*, thus

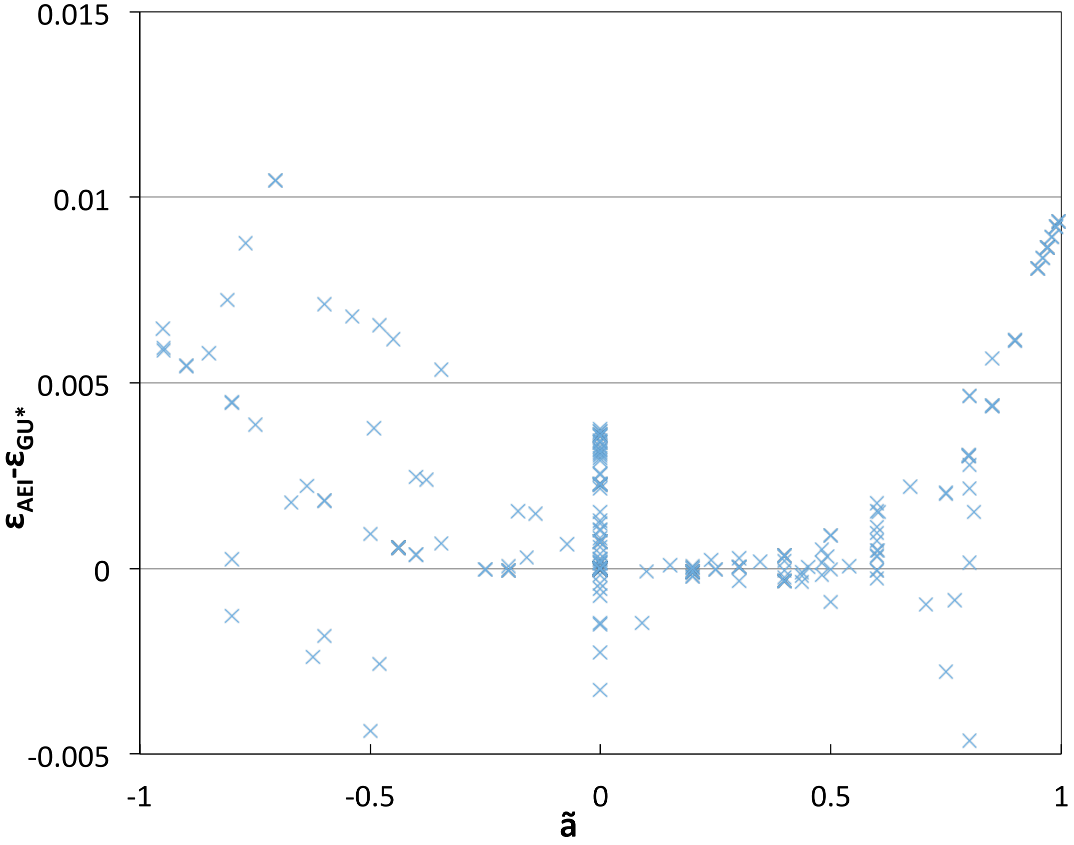
$$|\varepsilon_i| \leq 0.01$$



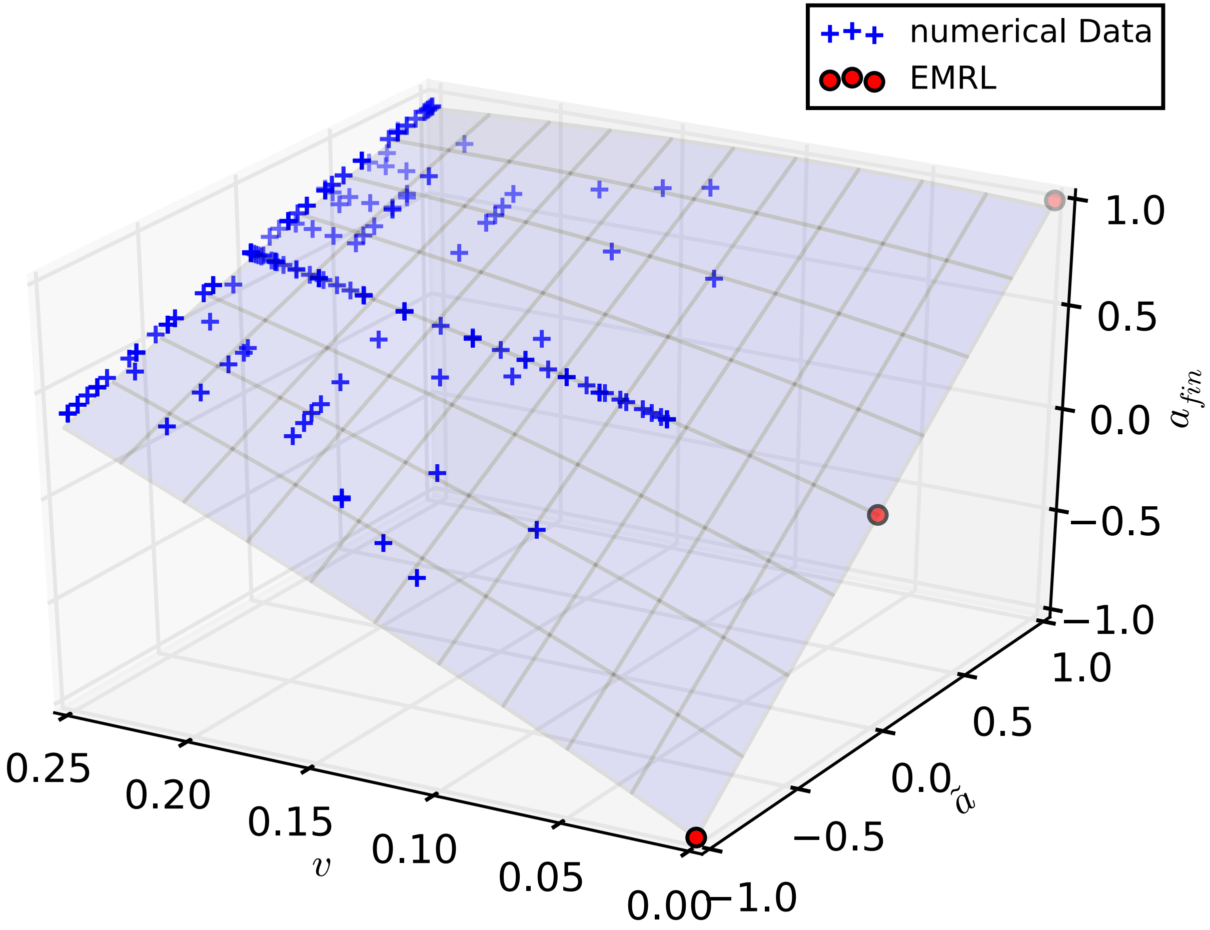
Sample fit:	GU	GU*
s4	-0.128	-0.208
s5	-0.347	
s7		-0.0415
s8		0.344
s9		-0.511
t0	-2.713	-2.742
t3	4.014	9.887
t4		-14.352

Additional terms reduce standard deviation by 15%


Histogram of residuals for the sample
AEI: $\mu = -0.002$, $\sigma = 0.0048$
GU*: $\mu = 0.0015$, $\sigma = 0.0026$
Count
$a_{fin}^{num} - a_{fin}^{fit}$
160
140
120
100
80
60
40
20
0
−0.025
−0.020
−0.015
−0.010
−0.005
0.000
0.005
0.010


$\varepsilon_{AEI}-\varepsilon_{GU^*}$
ã
0.015
0.01
0.005
0
-0.005
-1
-0.5
0
0.5
1

Distribution of the sampled data set?



Matching additional terms for generic expression

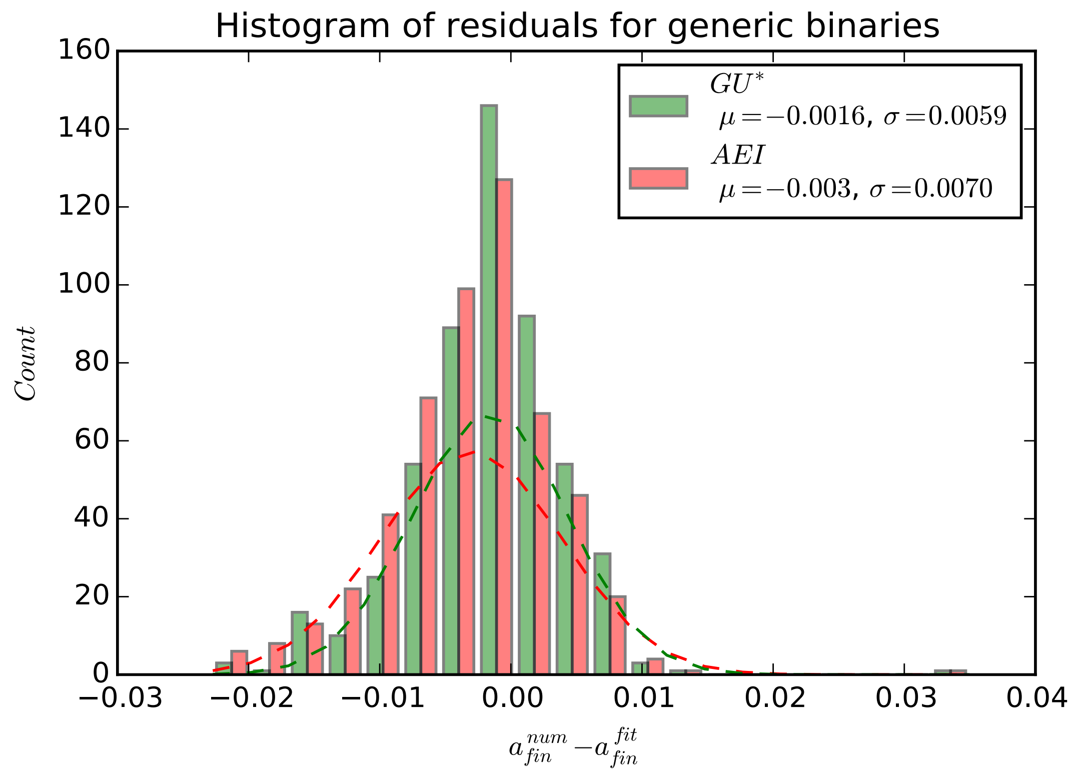
$$a_{fin} = \tilde{a} + \tilde{a}v(s_4\tilde{a} + s_5 v + s_7\tilde{a}^2 + s_8\tilde{a}v + s_9 v^2 + t_0) + 2\sqrt{3}v + t_2 v^2 + t_3 v^3 + t_4 v^4$$

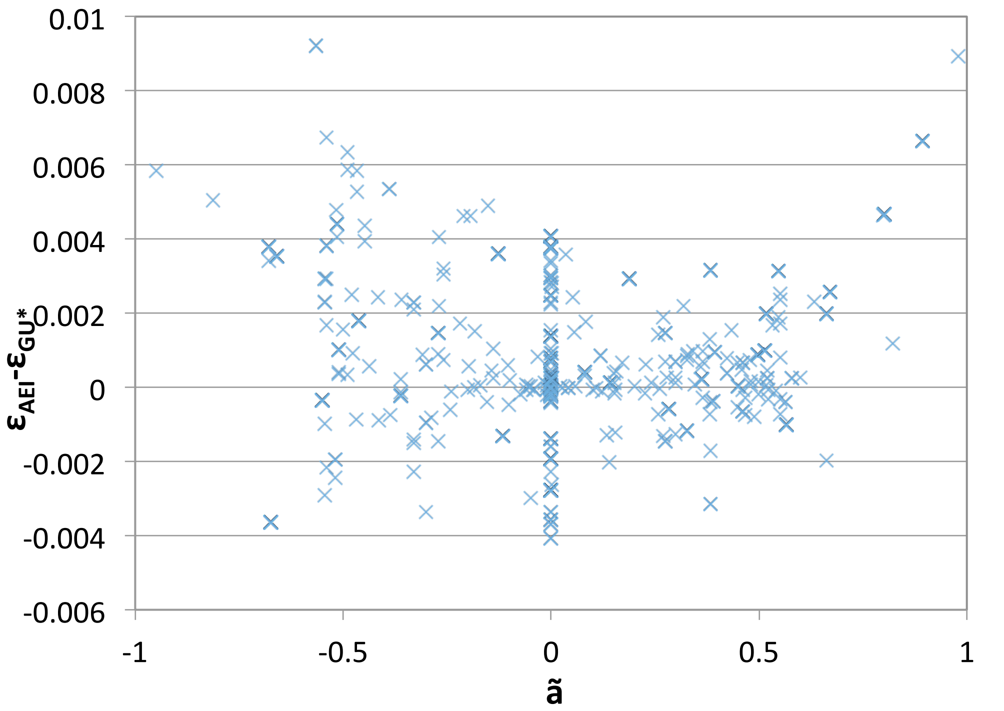
$$|\boldsymbol{a}_{fin}| = \frac{1}{(1+q)^2}(|\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2 q^4 + 2|\boldsymbol{a}_1||\boldsymbol{a}_2|q^2\cos\alpha + 2(|\boldsymbol{a}_1|\cos\tilde{\beta}(r_{In}) + |\boldsymbol{a}_2|q^2\cos\tilde{\gamma}(r_{In}))\underline{|\boldsymbol{l}|}q + \underline{|\boldsymbol{l}|^2}q^2)^{1/2}$$

$$\begin{aligned}|\boldsymbol{l}| &= 2\sqrt{3} + t_2 v + t_3 v^2 + t_4 v^3 \\ &+ \left(\frac{s_4 + s_8 v}{(1+q^2)^2} + \frac{s_7(|\boldsymbol{a}_1|\cos\tilde{\beta} + |\boldsymbol{a}_2|q^2\cos\tilde{\gamma})}{(1+q^2)^3}\right)(|\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2 q^4 + 2|\boldsymbol{a}_1||\boldsymbol{a}_2|q^2) \\ &+ \frac{s_9 v^2 + t_0 + 2}{1+q^2}(|\boldsymbol{a}_1|\cos\tilde{\beta} + |\boldsymbol{a}_2|q^2\cos\tilde{\gamma}).\end{aligned}$$

Does the GU* formula really improve the AEI formula? Validation?

Final spin norm



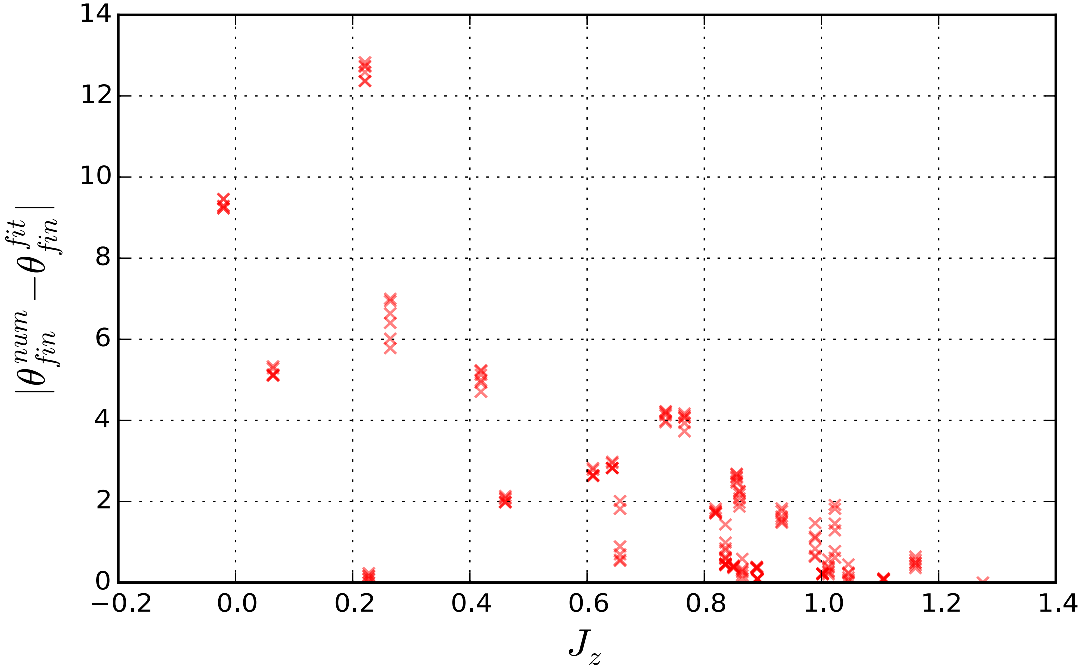


Note that $\boldsymbol{a} = \boldsymbol{L} \cdot (\boldsymbol{a}_1 + \boldsymbol{a}_2 q^2)/(1+q^2)$

Error analysis – final spin direction

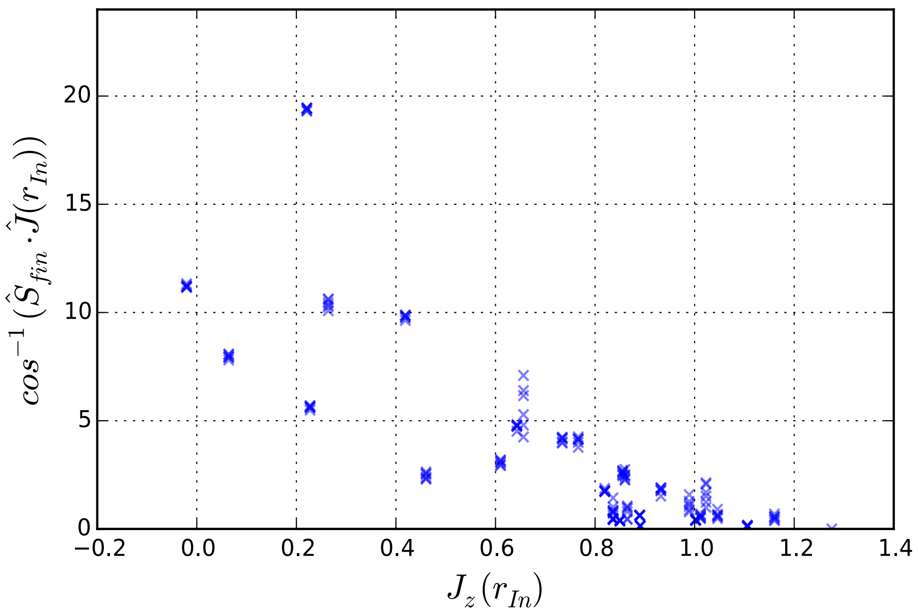
No significant correlation between the error of the final spin norm prediction and any quantity of the initial data, such as v, $\tilde{a}$ or $|\boldsymbol{a}_1 \times \boldsymbol{a}_2|$.

However, consider final spin direction:

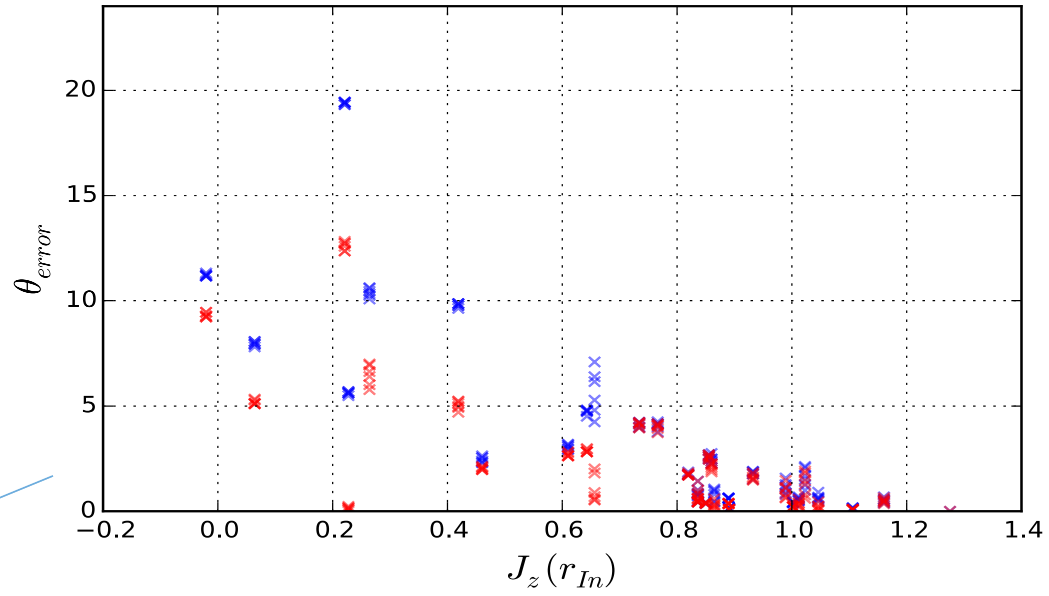


Only 172 simulations in the validation set provided the final spin direction

Validation test of assumption III: $\boldsymbol{S}_{fin} \| \boldsymbol{J}(r_{In})$



The lower the aligned component of the total angular momentum, the more imprecise is assumption III, adiabatic approximation is not valid
→ *transitional precession*



Final spin direction is partial affected by the defective generalization in assumption III
Solution?

Conclusion

- Improvement of the AEI successfully achieved consedring final spin norm: standard deviation reduced by nearly 50% for aligned BBH and 15% for generic BBH
- Additional terms (GU* formula) especially useful for BBH with aligned spin
- Validity for extreme mass ratios q = 0.01
- Final spin direction still impaired for $J_z < 0.5$

References

- http://www.black-holes.org/the-science-numerical-relativity/numerical-relativity/gravitational-lensing
- Bachelor Thesis, Fabian Hofmann