UNIVERSITY OF ICELAND

Entanglement Entropy at Non-Equilibrium in Holography

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Based on ArXiv: **1705.04696**, in collaboration with

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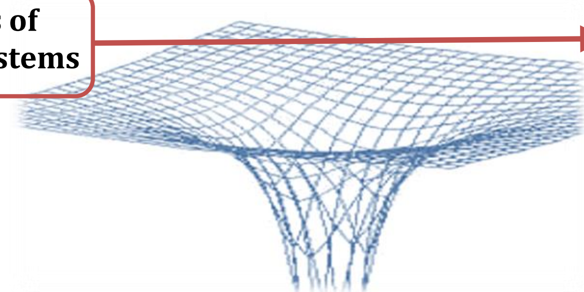
Context

Real time dynamics of strongly correlated systems



- Directly computable
- Easy collective responses

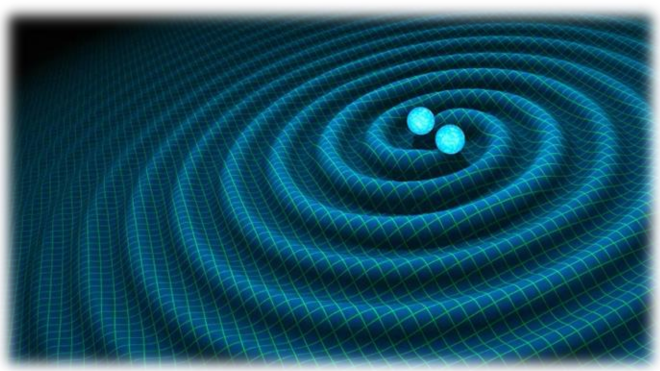
❖ Quark-gluon plasma thermalization *[Chesler, Yaffe, Heller, Romatschke, Mateos, van der Schee]*

❖ Quantum quenches *[Balasubramanian, Buchel, Myers, van Niekerk, Das]*

❖ Driven superconductors *[Rangamani, Rozali, Wong]*

→ *Important conclusion:*

Transition to hydrodynamic regime occurs very early!



❖ Turbulence in Gravity *[Lehner, Green, Yang, Zimmerman, Chesler, Adams, Liu]*

→ Insight into gravity gained from high-energy physics

Motivation

Emergent Collective Behavior: Quantum effects ↔ Out of equilibrium physics

Context: Quantum Mechanics of many-body systems

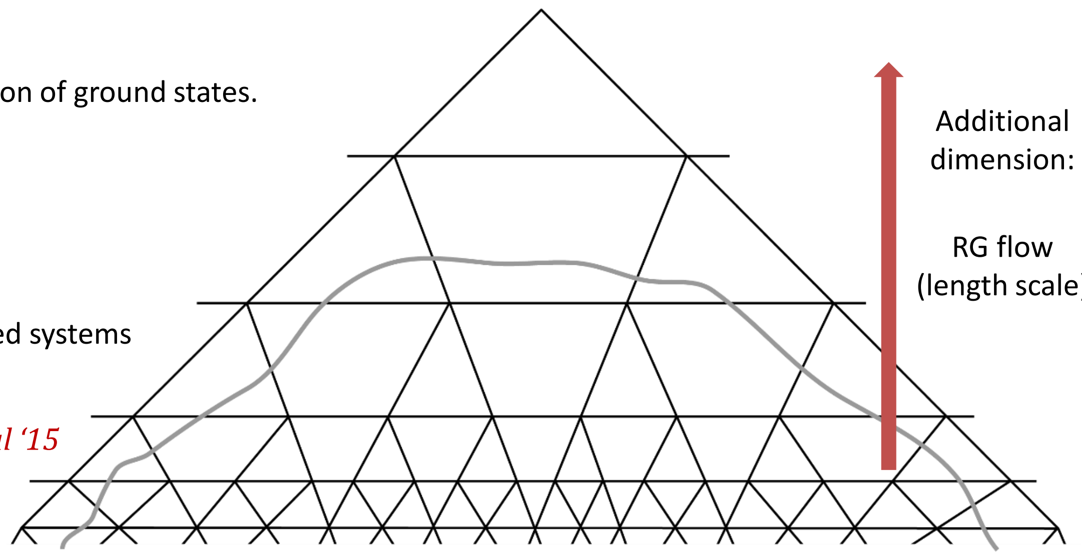
→ *How can we make predictions?*

1) **Entanglement**: Indicates structure of global wave function.
2) **RG group**: Increasing length scale, a sequence of effective descriptions is obtained.
3) **Entanglement Renormalization**: Careful removal of short-range entanglement.

4) **Tensor Networks**: Effective description of ground states.

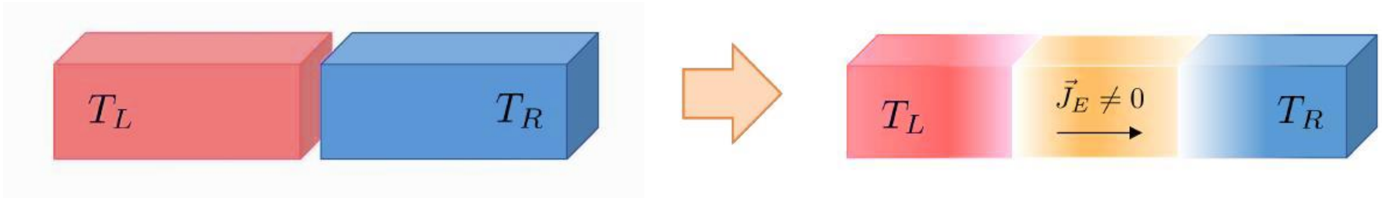
↓

Analysis of entanglement
to ascertain spatial structure of strongly coupled systems



Evenbly, Vidal '15

Setup Explanation

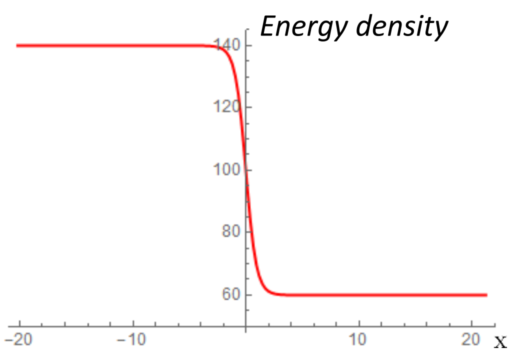


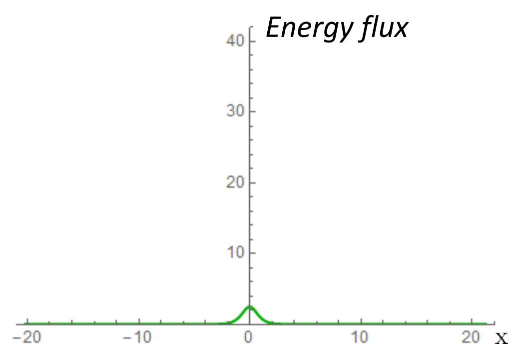
Bernard, Doyon '12
Chang, Karch, Yarom '13

- ***Initial configuration:***

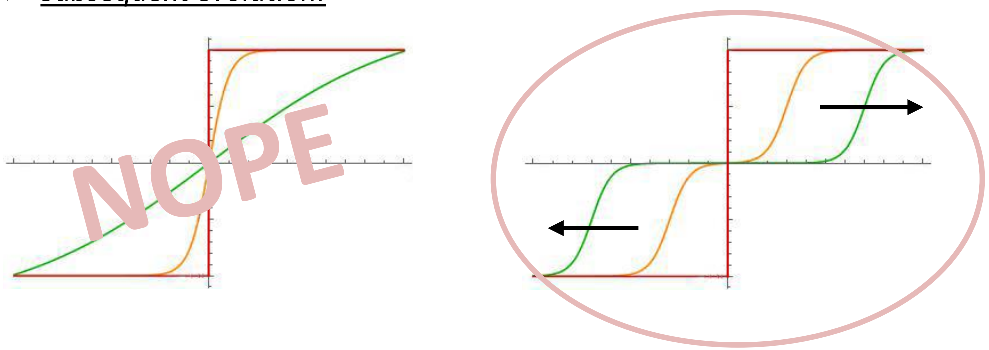
1+1 dimensional system separated into **two** regions,
independently prepared in thermal equilibrium.

$$T(t=0,x) = T_{\mathrm{L}}\,\theta(-x) + T_{\mathrm{R}}\,\theta(x)$$





- ***Subsequent evolution:***



A growing region with a constant energy flow, the **steady state**, develops.
This region is described by a thermal distribution at shifted temperature.
The state carries a constant energy current.

History Review

Bernard, Doyon '12

Thermal quench in 1+1

Two exact copies initially at equilibrium, independently thermalized.

Conservation equations & tracelessness:

$$\partial_x \langle T^{xx} \rangle = -\partial_t \langle T^{tx} \rangle = 0$$
$$\langle T^{xx} \rangle = \langle T^{tt} \rangle$$

$$\begin{cases} \langle T^{tx} \rangle = F(x-t) - F(x+t) \\ \langle T^{tt} \rangle = F(x-t) + F(x+t) \end{cases}$$

Expectation for CFT:

Shock waves emanating from interface, converge to non-equilibrium *Steady State*.

Bhaseen, Doyon, Lucas, Schalm '13

Generalization to any d

- Assume ctant. homogeneous heat flow as well:

$$\langle T^{\mu\nu} \rangle = a_d \, T^{d+1} \left(\eta^{\mu\nu} + (d+1) u^\mu u^\nu \right)$$

- Effective dimension reduction to 1+1.
- Linear response regime:

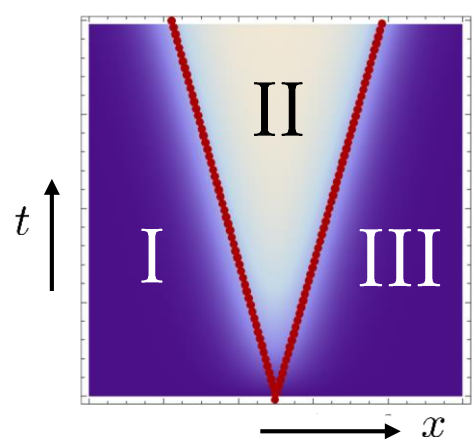
$$|T_L - T_R| << T_L + T_R$$

→ Hydro eqs. explicitly solvable.

Bhaseen, Doyon, Lucas, Schalm '13
Chang, Karch, Yarom '13

Hydrodynamical evolution of 3 regions

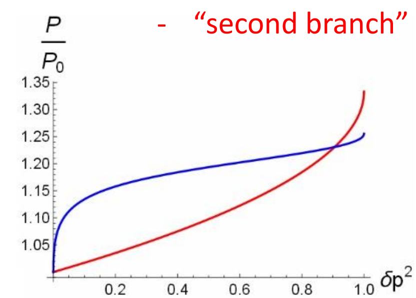
Match solutions ↔ Asymptotics of the central region.



$$T^{\mu\nu} = P(d\, u^\mu u^\nu + \eta^{\mu\nu}) + \pi^{\mu\nu} + \mathcal{O}(\partial^2)$$

Two configurations:

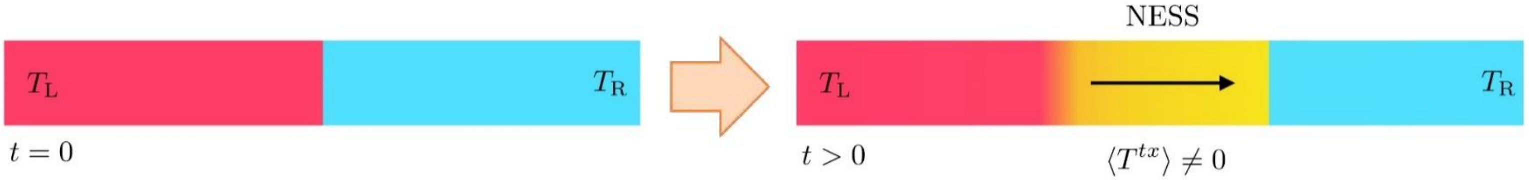
- Thermodynamic branch
- "second branch"



Rarefaction Wave

- There is no uniqueness of solution to the non-linear PDEs.
  - Doble shock solution: *Mathematically* correct, but not *physical*.
  - New solution: shock + rarefaction.

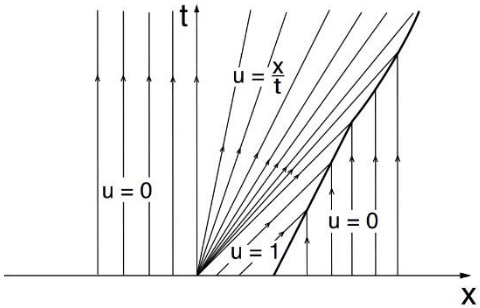
Spillane, Herzog '15
Lucas, Schalm, Doyon, Bhaseen '15
Hartnoll, Lucas, Sachdev '16



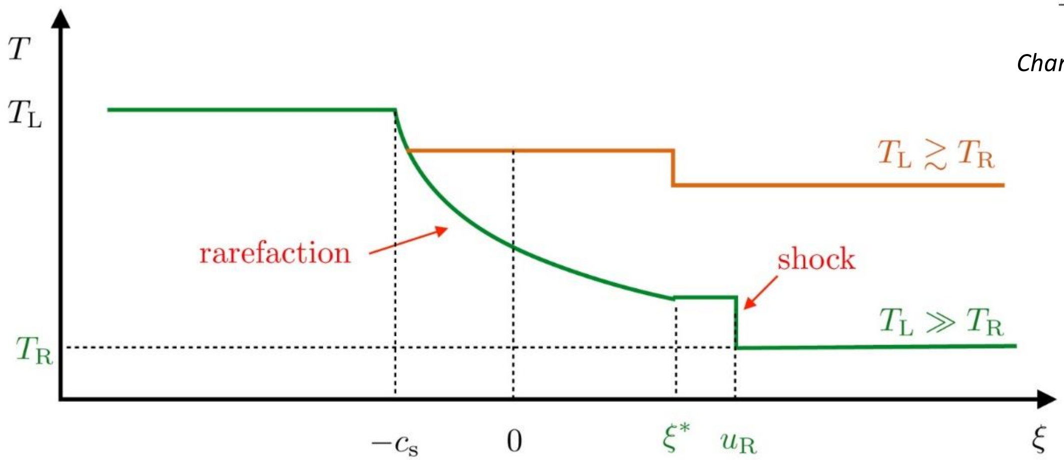
Entropy condition

Riemann problem: When we have conservation equations like $\partial_t u + \partial_x (f(u)) = 0$, the curves along which the initial condition is transported must end on the shock wave.

- The speed of the solution must be $f'(u_L) > (u_L + u_R)/2 > f'(u_R)$, which rules out a shock moving into the hotter region.



Characteristics must end in the shockwave, not begin.



ENTANGLEMENT TSUNAMIS

Liu, Suh '13
Li, Wu, Wang, Yang '13

Context: A global quench leading to an AdS black hole as final state.

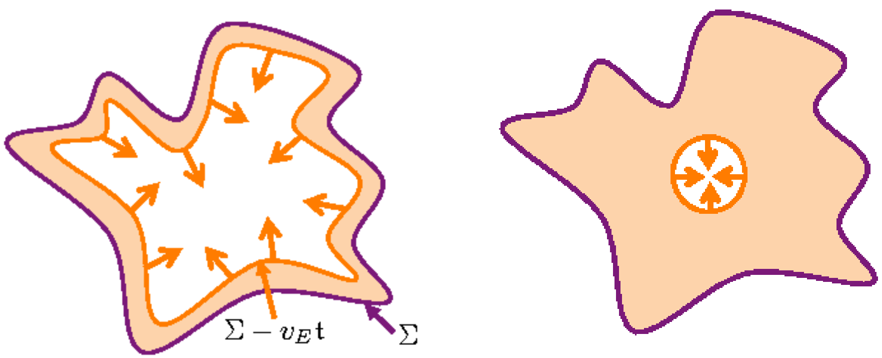
(Thin shell of matter which collapses to form a black hole)

$$ds^2 = \frac{L^2}{z^2}\left(-\left[1-\theta(t)\,g(z)\right]dt^2 - 2dt\,dz + d\vec{x}^2\right)$$

Entanglement growth: Initially quadratic, then followed by a universal linear regime.

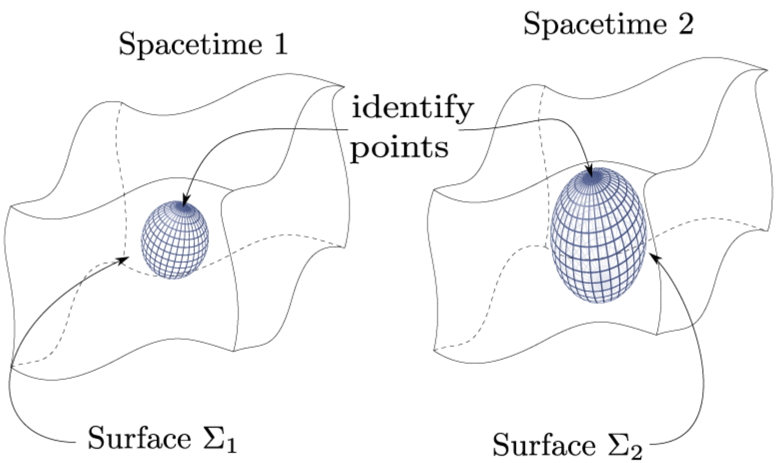
$$\Delta S_\Sigma(t) = \frac{\pi}{d-1}\mathcal{E}\,A_\Sigma t^2 + \ldots \quad \longrightarrow \quad \Delta S_\Sigma(t) = s_{\rm eq}\left(V_\Sigma - V_{\Sigma - v_E t}\right)t + \ldots$$

Simple geometric picture: A wave with a sharp wave-front propagating inward from Σ, and the region that has been covered by the wave is entangled with the region outside Σ, while the region yet to be covered is not so entangled.



Gluing Spacetimes

Israel '66



Induced metric $\gamma_1 = \gamma_2$

Take two spacetimes and define codimension one hypersurfaces $\Sigma_{1/2}$ such that they have the same topology.

If the induced metric on $\Sigma_{1/2}$ is the same ($\gamma_1 = \gamma_2 = \gamma$), the two spacetimes can be matched by identifying $\Sigma_{1/2}$ if the energy-momentum on Σ satisfies

$$\left(K^{+}_{ij} - \gamma_{ij} K^{+}\right) - \left(K^{-}_{ij} - \gamma_{ij} K^{-}\right) = -\kappa S_{ij}$$

- Israel Junction Conditions -

S_{ij} : Energy momentum tensor on the surface
γ_{ij} : Induced metric
$K^{+,-}$: Extrinsic curvatures depending on embedding.

In our setup:

$$ds^2 = \begin{cases} ds^2_{T_L} & \text{if } x < -t \\ ds^2_{\text{boost}} & \text{if } \; -t < x < t \\ ds^2_{T_R} & \text{if } x > t \end{cases}$$

Discontinuous geometry!

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + g_{\mu\nu}(z,x,t)dx^\mu dx^\nu\right)$$

There is an analytic solution…

$$g_{tt}(z,x,t) = -\left[1 - \frac{z^2}{L^2}\left(f_R(x-t) + f_L(x+t)\right)\right]^2 + \left[\frac{z^2}{L^2}\left(f_R(x-t) - f_L(x+t)\right)\right]^2$$

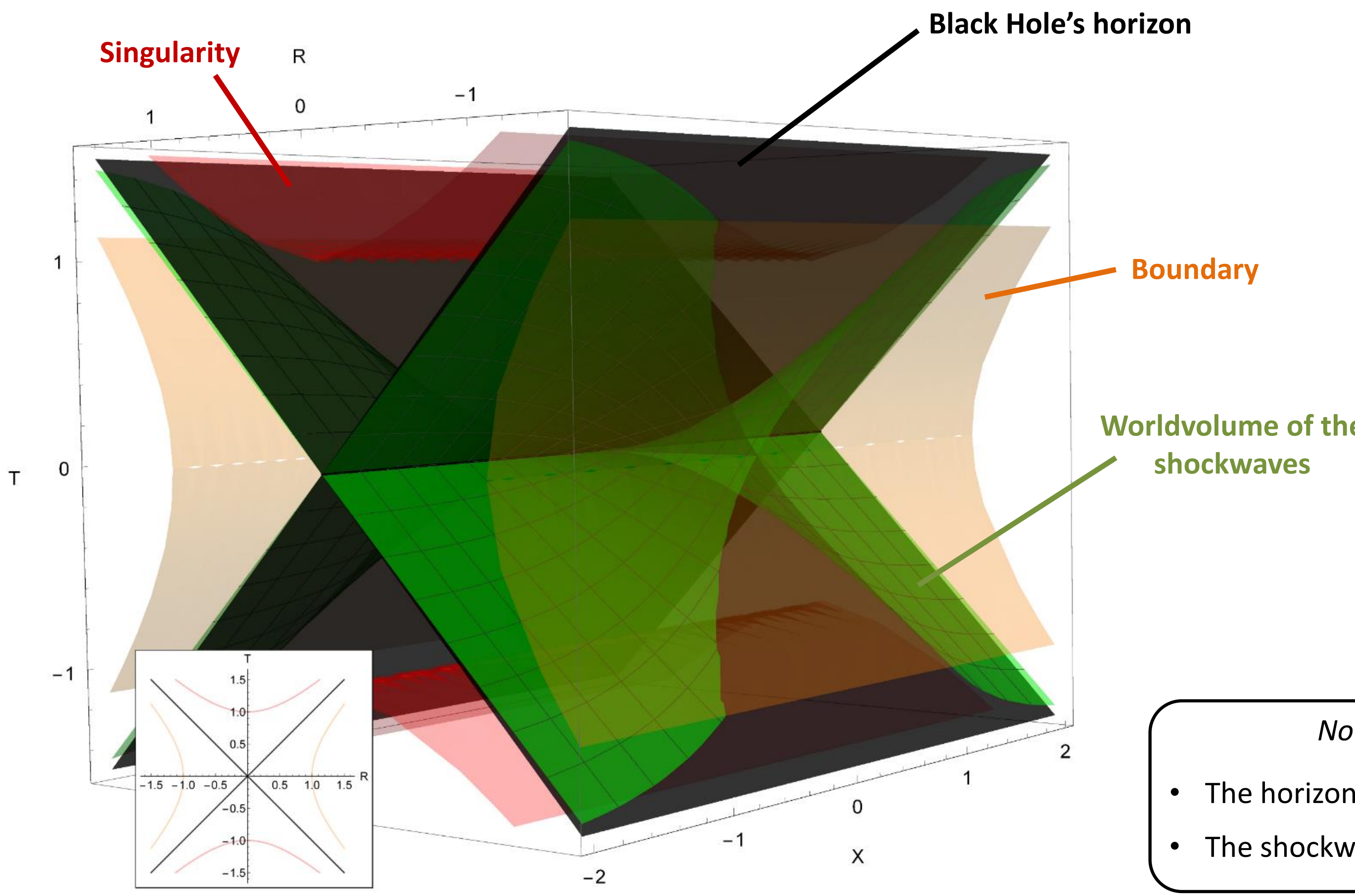
with
Initial condition: $f_{L/R}(v) \to \frac{\pi^2 L^2}{2}\left(T_L^2 + \left(T_R^2 - T_L^2\right)\theta(v)\right)$

But… is the horizon cut into 3 pieces??

Spacetime Diagram

Coordinates compactified:

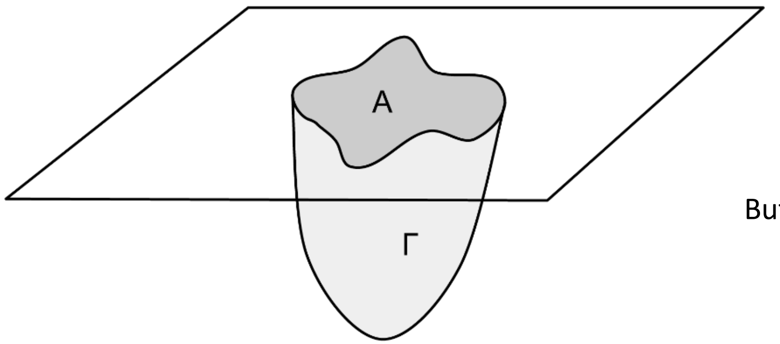
$$T = \left|\frac{z - z_H}{z + z_H}\right|^{1/2} \sinh\left(\frac{t}{z_H}\right), \quad R = \left|\frac{z - z_H}{z + z_H}\right|^{1/2} \cosh\left(\frac{t}{z_H}\right)$$



Notice that:

- The horizon remains untouched
- The shockwaves are spacelike

ENTANGLEMENT ENTROPY



The geometry is discontinuous:

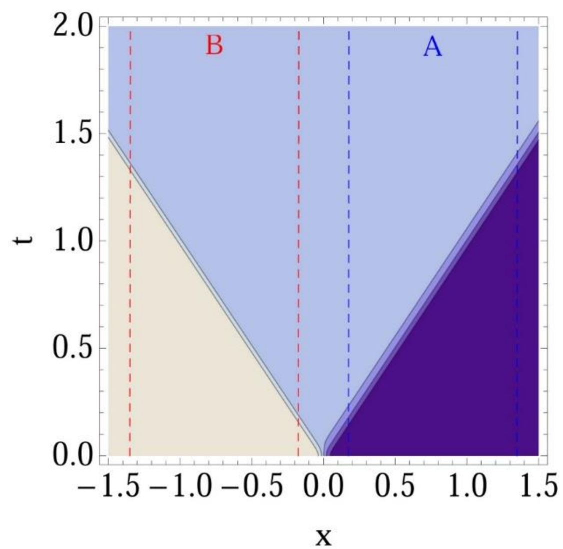
$$f_{L/R}(v) \to \frac{\pi^2 L^2}{2}\left(T_L^2 + \left(T_R^2 - T_L^2\right)\theta(v)\right)$$

But we replace the step function by this initial condition

$$f_L(v) = f_R(v) = \frac{\pi^2 L^2}{4}\Big(\left(T_L^2 + T_R^2\right) + \left(T_R^2 - T_L^2\right)\tanh(\alpha v)\Big)$$

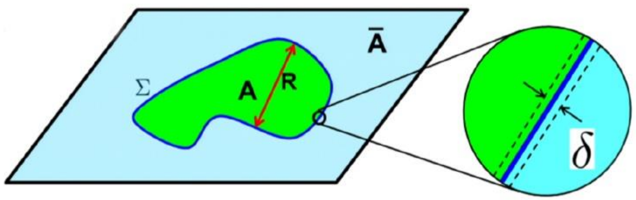
➢ **Shooting method to find geodesic lengths**: Shoot from the tip until the desired boundary values are obtained.

Intervals A, B considered:



Contour plot of the energy density

Entanglement Regularization:

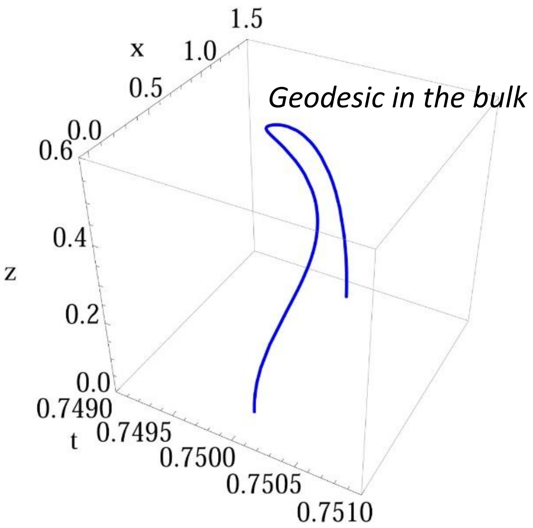


- Small distance contributions must be substracted
- We use minimal substraction scheme:

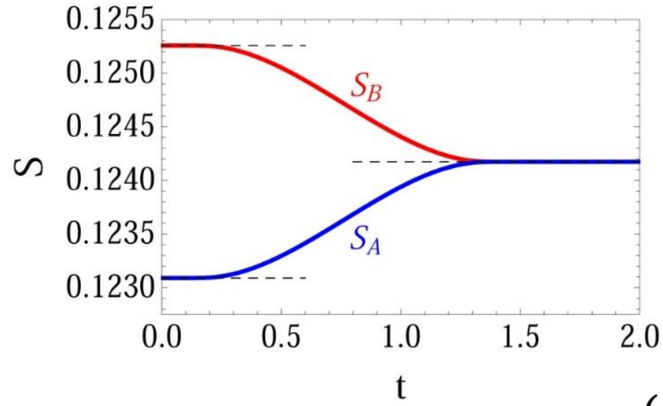
$$S_A^{\text{ren}} = \frac{1}{4G}\left(\text{Area}(\gamma_A) - \text{Area}(\gamma_A^{div})\right)$$

with $\text{Area}(\gamma_A^{div}) = -2L\log\delta$

Universal Law



Time evolution of the entanglement entropy of intervals A and B:



$$S_A(t) = \begin{cases} S_A(t=0) & 0 \leq t \leq t_1 \\ S_A(t) & t_1 \leq t \leq t_2 \\ S_A(t=\infty) & t_2 \leq t \end{cases}$$

Define the normalized entanglement entropy:

$$f_A(\rho) \equiv \frac{S_A(t) - S_A(t=0)}{S_A(t=\infty) - S_A(t=0)}$$

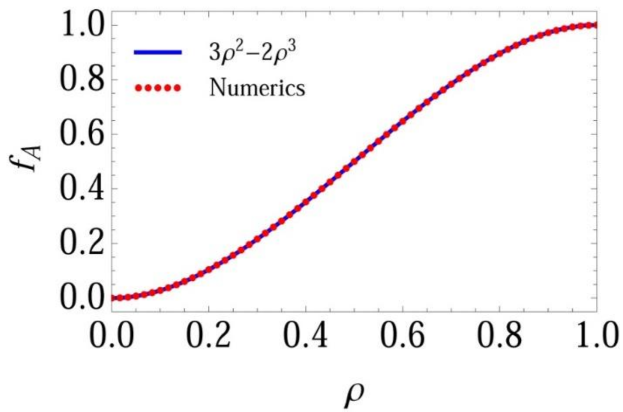
where

$$\rho \equiv (t - t_1)/\ell$$

Plots overlay on top of each other,

numerical behavior seems to be well approximated by

$$f_A(\rho) \simeq 3\rho^2 - 2\rho^3\,, \qquad 0 \leq \rho \leq 1$$



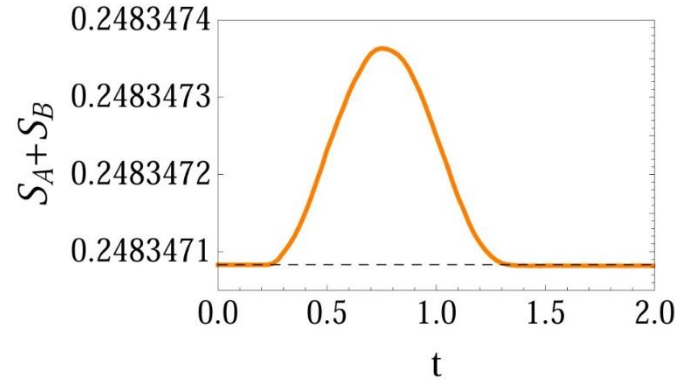
Non – Universal Effects

❖ Conservation of entropy:

$$S_A(t=0) + S_B(t=0) = S_A(t=\infty) + S_B(t=\infty)$$

But it's not conserved at intermediate times!

$$S_{A+B}(t) \neq const$$



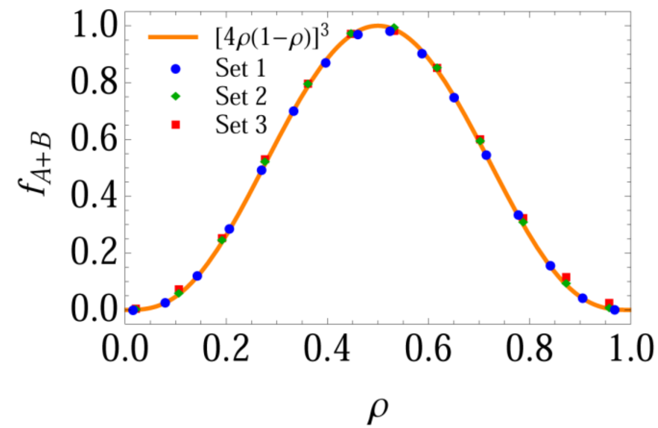
❖ Define the normalized total entanglement entropy:

$$f_{A+B}(\rho) \equiv \frac{S_{A+B}(t) - S_{A+B}(t=0)}{S_{A+B}(t_{\max}) - S_{A+B}(t=0)} \quad \text{where} \quad \rho \equiv (t - t_1)/\ell$$

Plots overlay on top of each other,

numerical behavior seems to be well approximated by

$$f_{A+B}(\rho) \simeq [4\rho(1-\rho)]^3 \,, \qquad 0 \leq \rho \leq 1$$



❖ *Conclusion:* Non-conservation effects are caused by **non-universal contribution**:

$$f_A(\rho) = 3\rho^2 - 2\rho^3 + \boxed{C(T_L, T_R, \ell)}[4\rho(1-\rho)]^3$$

Factor with non-universal dependence
on the parameters of the interval

Mutual Information

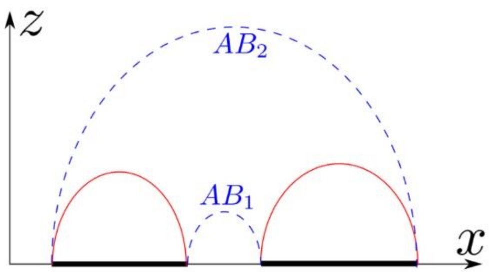
How does information get exchanged between the systems which are isolated at t=0?

Def.) $I(A,B) = S_A + S_B - S(A \cup B)$ where $S(A \cup B) = \min\left\{S_A + S_B, S_1 + S_2\right\}$

<u>*Interpretation:*</u>

It measures which information of subsystem A is contained in subsystem B.

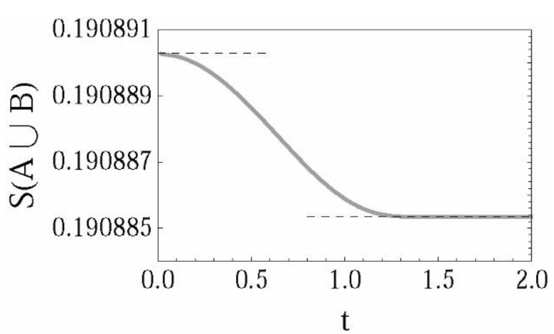
In other words: The amount of information that can be obtained from one of the subsystems by looking at the other one.

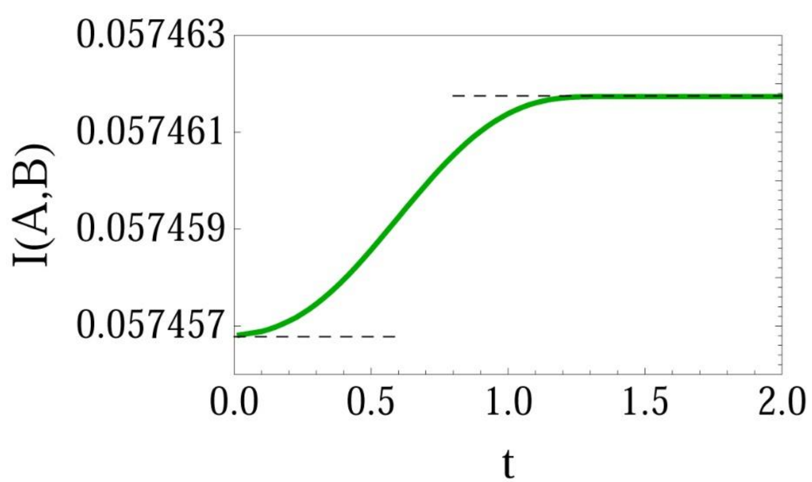


Note that $I(A,B) \geq 0$ always.

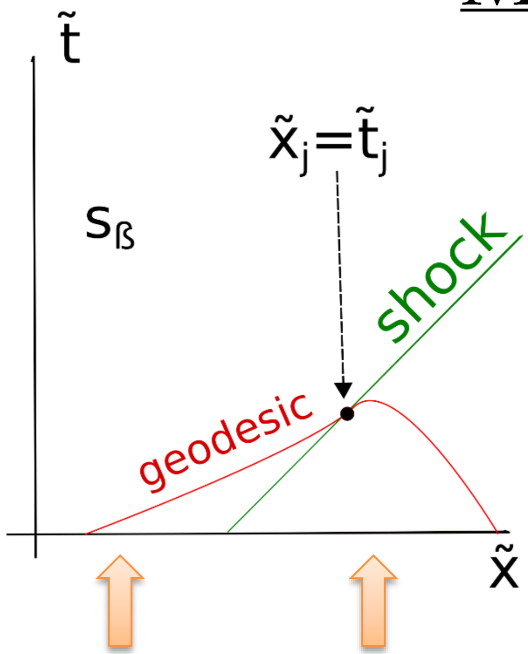
Observation: $\partial_t I(A,B) \geq 0$

- ❖ The shockwaves transport information about the presence of the other heat bath., although they are spacelike in the bulk.





Matching Geodesics



One end on the steady state, another in the thermal region.

Condition for the position of the shockwave:

$$x_j = t_j \Leftrightarrow \tilde{x}_j = \tilde{t}_j$$

Complementary approach – Steps:

1) Calculate geodesics in each spacetime region.
2) Add their renormalized lengths
3) Extremize the sum with respect to the meeting point.

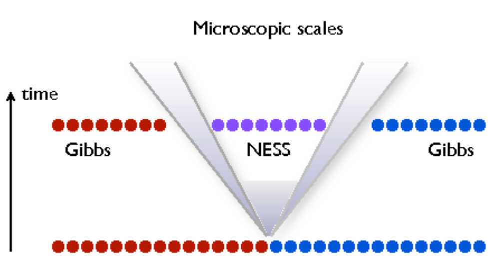
In previous calculations,

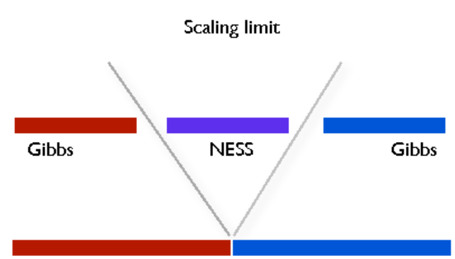
$$f_L(v) = f_R(v) = \frac{\pi^2 L^2}{4}\Big(\left(T_L^2 + T_R^2\right) + \left(T_R^2 - T_L^2\right)\tanh(\alpha v)\Big)$$

Here, the metric components are discontinuous

$$f_{L/R}(v) \to \frac{\pi^2 L^2}{2}\left(T_L^2 + \left(T_R^2 - T_L^2\right)\theta(v)\right)$$

→ Agreement between numerical results at large α and results from this approach?





Note:

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + g_{\mu\nu}(z,x,t)dx^\mu dx^\nu\right)$$

→ Schwarzchild coordinates!

Small Temperature Expansion

❖ In this limit, we can prove the previous universal law:

$$f_A(\rho) \simeq 3\rho^2 - 2\rho^3\,, \qquad 0 \leq \rho \leq 1$$

❖ The replacement is:

$$T_L \to \delta\, T_L, \quad T_R \to \delta\, T_R \quad \Longrightarrow \quad \begin{aligned} \partial_x d_R \propto \partial_{z_j} d_L &\propto (\ell - t)(\ell + 2t - 4x)t - (\ell - 2t) z_j^2, \\ \partial_{z_j} d_R \propto \partial_x d_L &\propto (\ell - t)(t - 2x)(\ell + t - 2x)t + z_j^4. \end{aligned}$$

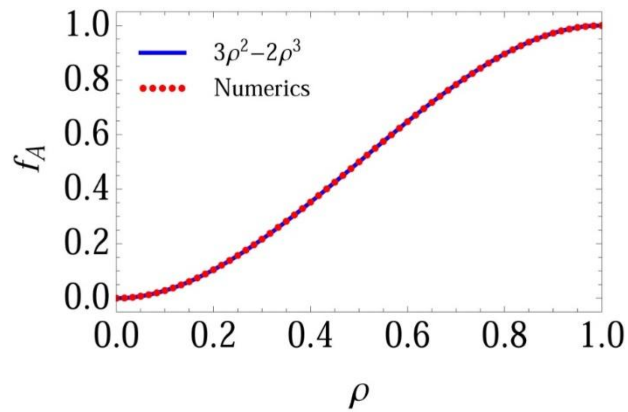
extremized for $z_j = t\sqrt{\ell - t}$

❖ **Quasiparticle description:**

Low-energy spectrum of excitations of some systems are governed by effectively conformal theories,

when <u>both temperatures are low</u>. *Bernard, Doyon '16*

…so the highest lying parts of the spectrum are not populated.



➢ Universal formula should be valid in ballistic regimes of actual electronic systems. Correlation functions too? Lattice model expectations?

$$\begin{aligned} d_R(z_j, x) = &\log\left[(1 + \pi^2 T_R^2 \tilde{z}_j^2)\cosh(2\pi T_R(x - \ell)) - (1 - (\pi T_R \tilde{z}_j)^2)\cosh(2\pi T_R(t - x))\right] + \\ &\log\Big[\,(1 + \pi^2 T_L T_R \tilde{z}_j^2)\cosh(\pi(tT_L - tT_R + 2T_R x)) \\ &+ (\pi^2 T_L T_R \tilde{z}_j^2 - 1)\cosh(\pi(t(T_L + T_R) - 2T_R x))\Big] - \frac{1}{2}\log\left(16\pi^8 T_L^2 T_R^6 \tilde{z}_j^4\right) \end{aligned}$$

Matching Geodesics - Results

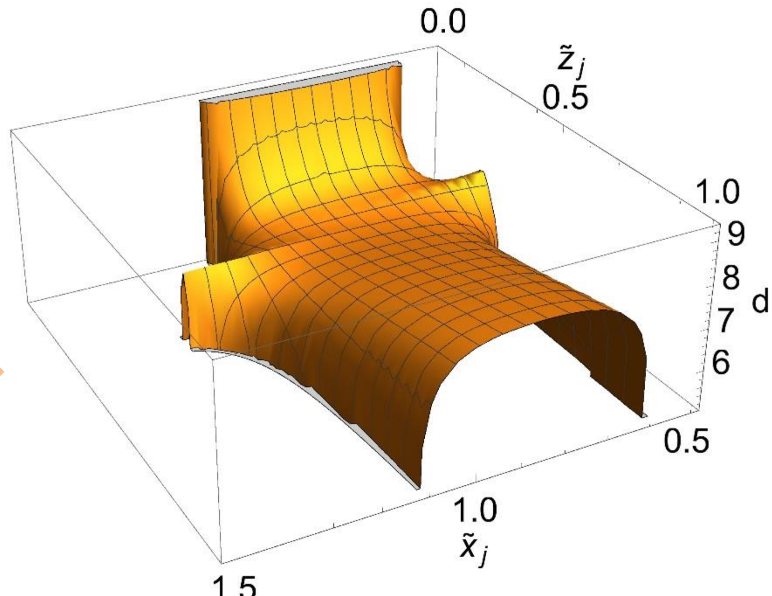
- Extremization:

$$\partial_{\tilde{z}_j} d_R = 0\,, \quad \partial_x d_R = 0$$

→ Numerical methods to solve non-linear algebraic equations.

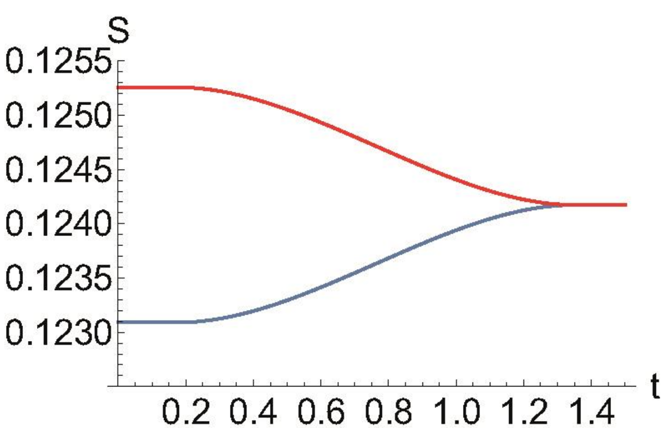
- The distance function turns imaginary outside of some region.

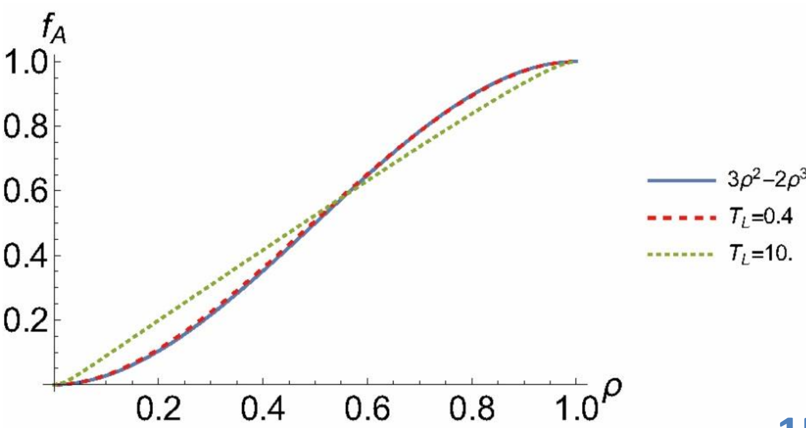
(if one boundary point becomes null or timelike-separated from the joining point)



Distance function $d_R(z_j, x)$

- Argument from Kruskal diagram → Exclude solutions with $\tilde{z}_j > \tilde{z}_H$





VELOCITY IN ENTANGLEMENT GROWTH

Liu, Suh '13
Li, Wu, Wang, Yang '13
Hartman, Maldacena '13

- After a global quench, the entanglement entropy exhibits quadratic growth:

$$\Delta S(t) \propto t^2 + \dots$$

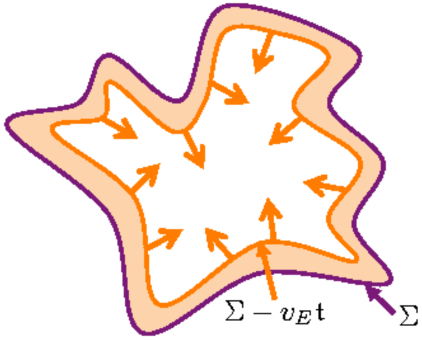
- Followed by a universal linear growth regime where

$$\Delta S(t) = v_E s_{eq} A_\Sigma t + \dots$$

- The velocity v_E depends on the final equilibrium state. In the case of an AdS-RN black hole,

$$v_E = \frac{\sqrt{d}(d-2)^{\frac{1}{2}-\frac{1}{d}}}{(2(d-1))^{1-\frac{1}{d}}}$$

← **Tsunami Velocity**



- **Butterfly velocity**: Speed of propagation of chaotic behavior in the boundary theory:

$$v_B = \sqrt{\frac{d}{2(d-1)}}$$

Shenker, Stanford '13
Roberts, Stanford, Susskind '14

$$W_x(t) = e^{-iHt} W_x e^{iHt}$$

For an operator local on the thermal scale, defined on a Tensor Network

Bound between these velocities:

$$1 \geq v_B \geq v_E$$

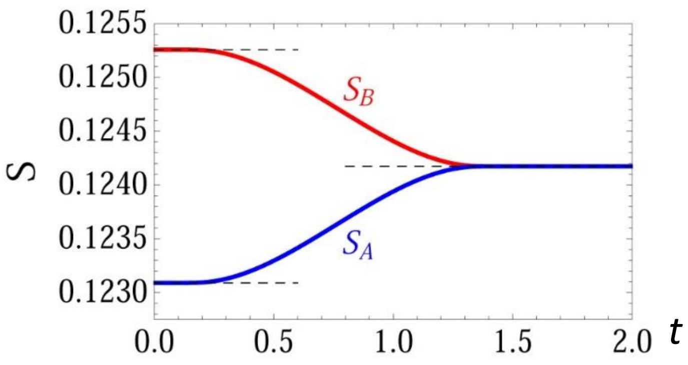
Bounds in Velocities

Rangamani, Rozali, Vincart-Emard '17

- **Average Velocity**

Average entropy increase rate:

$$v_{av} \equiv \frac{\Delta S}{\Delta t} = \frac{L}{4G\ell} \log\left(\frac{T_L \sinh(\pi \ell T_R)}{T_R \sinh(\pi \ell T_L)}\right)$$



Constant – Evolve - Constant

This quantity is bounded, although it can be arbitrarily large:

$$\lim_{\ell \to \infty} v_{av} = \frac{L}{4G} \pi |T_R - T_L|$$

Normalized by the entropy density of the final state, we find

$$|\tilde{v}_{av}| \leq \left|\frac{T_R - T_L}{T_R + T_L}\right| \leq 1$$

To compare with $\Delta S(t) = v_E s_{eq} A_\Sigma t + \dots ,$

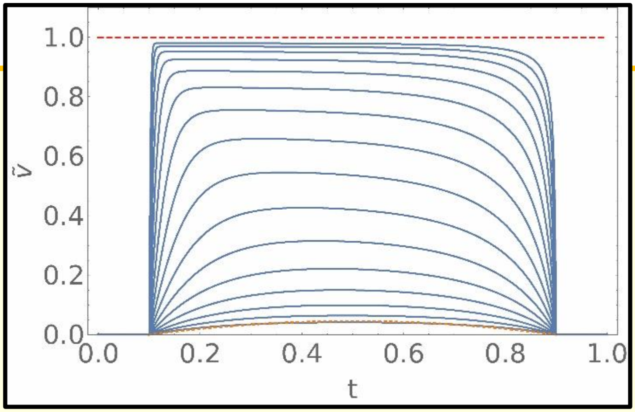
where $\tilde{v}_{av} \equiv v_{av}/s_{eq}$

- *When normalized in a **physical** way, we get a similar bound as 2d entanglement tsunamis or local quenches.*

- **Momentary Velocity**

$$\tilde{v} \equiv \frac{1}{s_{eq}} \frac{dS(\ell, t)}{dt} \leq 1$$

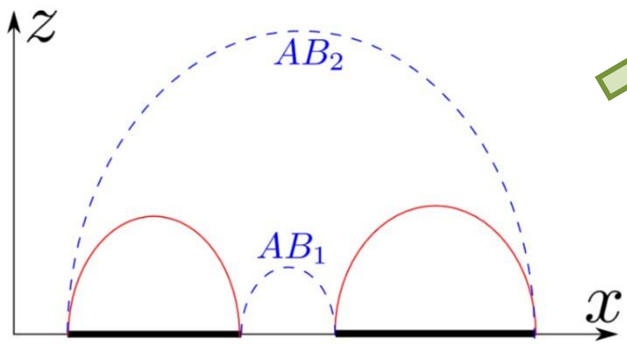
Numerically, we still find this bound.



- *Interpretation: The shockwave seems to take the role that the entanglement tsunami had for a global quench.*

n>2 Disconnected Intervals

Hubeny, Rangamani, Takayanagi '07



Two physical configurations for calculating the entanglement entropy.

- Choose the minimal possible configuration:

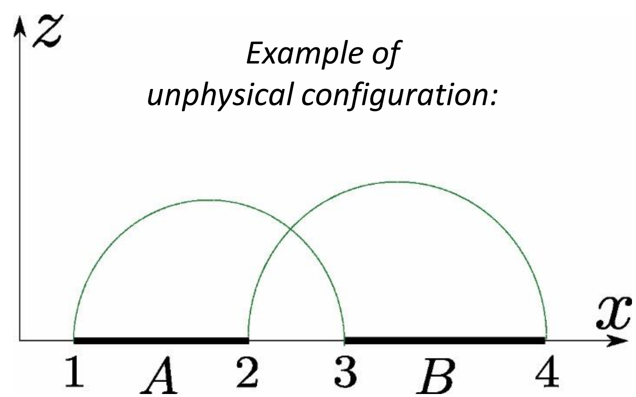
$$S(AB) = \min\{S(A)+S(B), S_{AB_1}+S_{AB_2}\}$$

Phase transitions! **Configurations = Phases**

Entanglement entropies are required to satisfy certain inequalities

- ✓ Subadditivity: $S(AB) \leq S(A)+S(B)$ *Araki, Lieb '70*
- ✓ Triangle: $S(AB) \geq |S(A)-S(B)|$

Example of unphysical configuration:



- Similar concepts with n>2 intervals? *Mirabi, Tanhayi, Vazirian '16* *Bao, Chatwin-Davies '16*

- When enumerating the possible phases, we must exclude those with curves intersecting (*unphysical* phases)

Physical Interval Phases

$$S(AB) = S(A) + S(B) \;\Leftrightarrow\; \begin{pmatrix} 1 \to 2 \\ 3 \to 4 \end{pmatrix}$$ "disconnected phase" 

$$S(AB) = S(AB_1) + S(AB_2) \;\Leftrightarrow\; \begin{pmatrix} 1 \to 4 \\ 2 \to 3 \end{pmatrix}$$ "connected phase" 

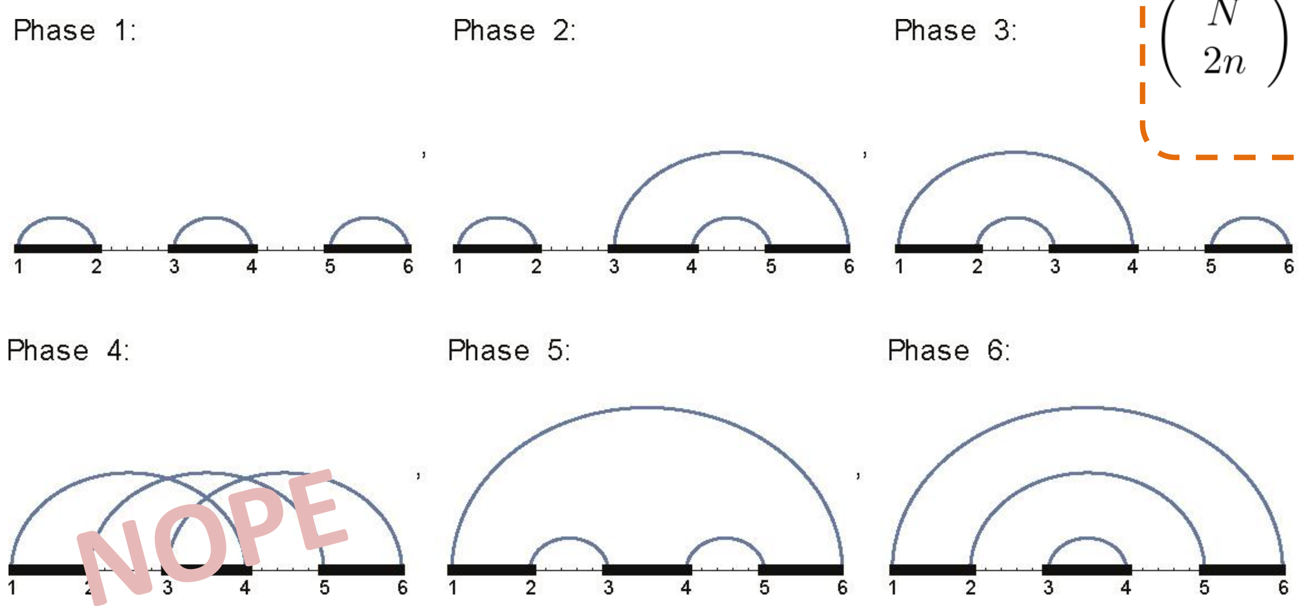
Headrick, Takayanagi '07
Hubeny, Maxfield, Rangamani, Tonni '13

Unphysical configurations

- Do not yield lowest values for the entanglement entropy.
- In a time-dependent case, the co-dimension one surface spanned would become null or timelike.

$$\begin{pmatrix} N \\ 2n \end{pmatrix} = \frac{N!}{(2n)!(N-2n)!}$$

ways to join intervals



Generalized Inequalities

n=3 case

- Strong Subadditivity inequality: *Lieb, Ruskai '73* ✓ Time dep. case

$$S(AB) + S(BC) - S(ABC) - S(B) \geq 0$$

- A different inequality, which was proven for the holographic prescription: *Headrick, Takayanagi '07*

$$S(AB) + S(BC) - S(A) - S(C) \geq 0$$

- Monogamy of mutual information == Negativity of tripartite information:

$$I_3(A:B:C) \equiv S(A)+S(B)+S(C)-S(AB)-S(BC)-S(AC)+S(ABC) \leq 0$$

A. Wall '12 ✓ Time dep. case

n>3 cases

- For n=5 intervals (A, B, C, D, E), this generalizes to 5 inequalities.
- Negativity of n-partite information:

$$I_n(A_1:A_2:A_3:...:A_n) \equiv \sum_{i=1}^{n} S(A_i) - \sum_{i<j}^{n} S(A_i \cup A_j) + \sum_{i<j<k}^{n} S(A_i \cup A_j \cup A_k)$$
$$\mp ... + (-1)^n S(A_1 \cup A_2 \cup ... \cup A_n),$$

- Proposed inequalities:

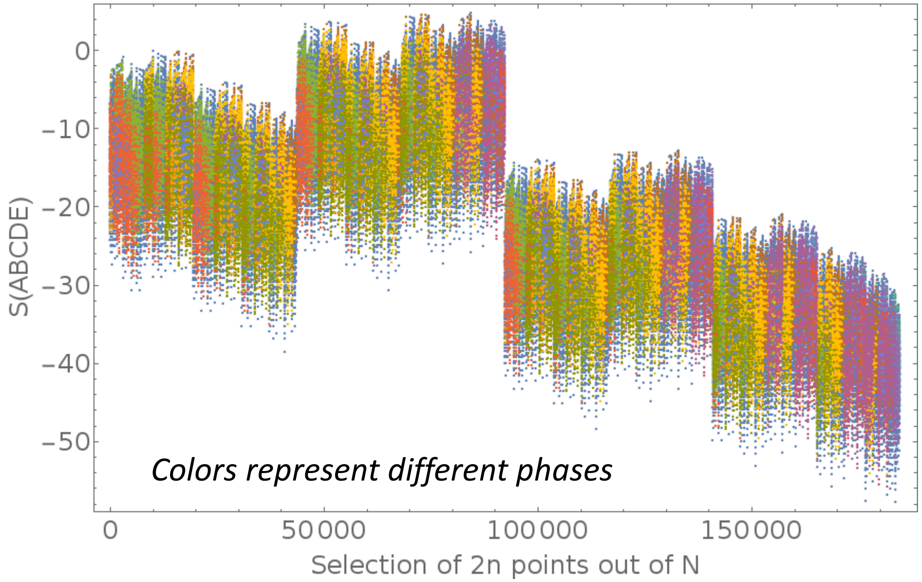
Alishahiha, Mozaffar, Tanhayi '14
Mirabi, Tanhayi, Vazirian '16

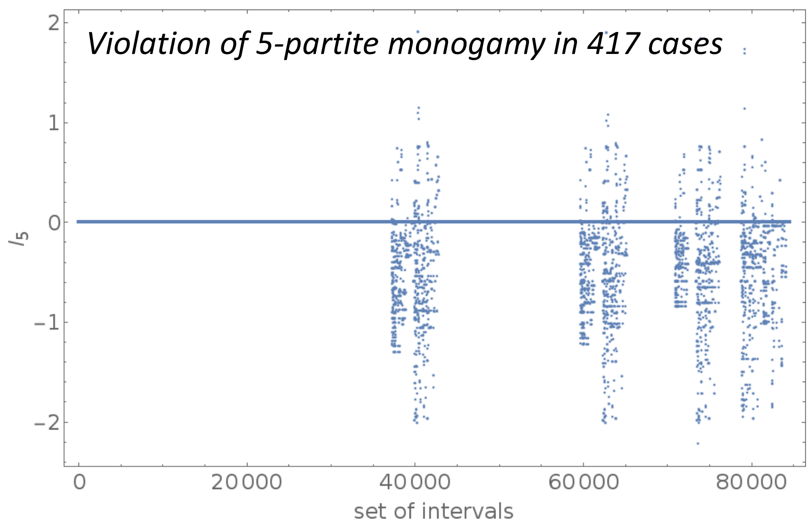
$$\begin{cases} I_4(A:B:C:D) \geq 0 \\ I_5(A:B:C:D:E) \leq 0 \end{cases}$$

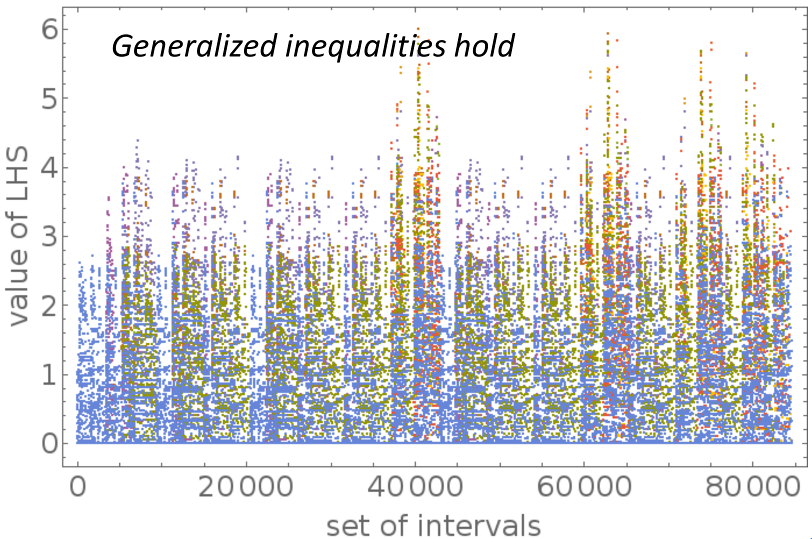
...which do **not** hold in holographic setups. *Hayden, Headrick, Maloney '11*

Results for n=5 Intervals

- 42 physical phases
- 20 boundary points
- 184756 possible unions
- 84579 not totally disconnected

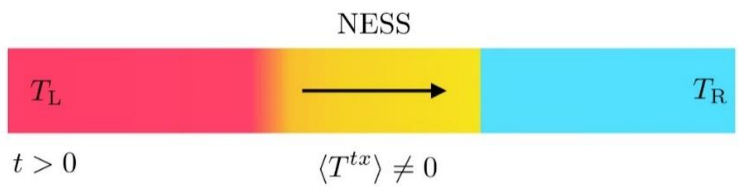






Higher Dimensions

- A solution was found in the hydrodynamic regime — *Bhaseen, Doyon, Lucas '15*
- A similar solution in the holographic setup confirmed it — *Amado, Yarom '15*
- An inconsistency between results and thermodynamics was found — *Spillane, Herzog '15*



- The higher-dimensional case is more physically relevant and interesting.

Assuming that the dual-shock solution is valid approximately:

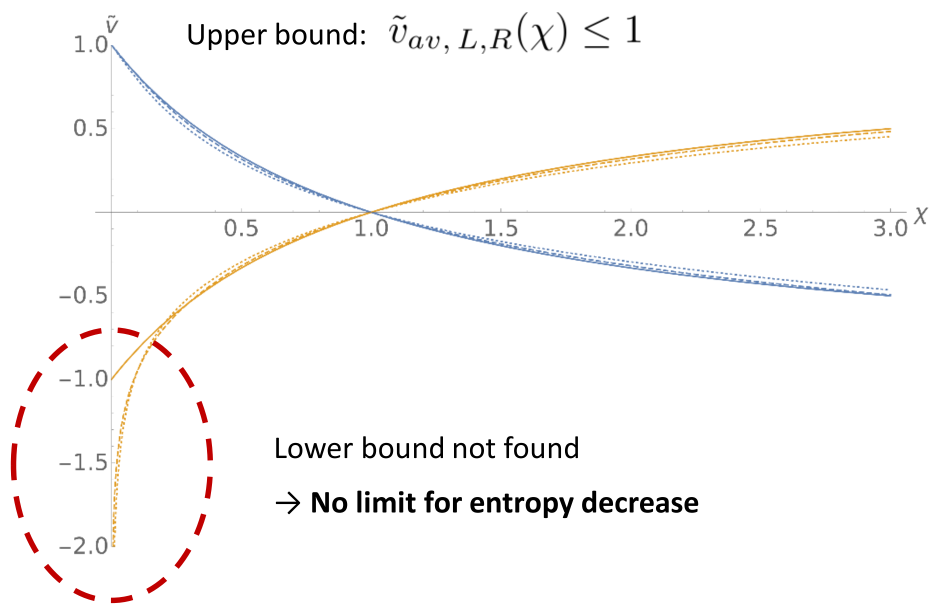
The shockwaves move with different velocities:

$$u_L = \frac{1}{d-1}\sqrt{\frac{\chi + d - 1}{\chi + \frac{1}{d-1}}}, \quad u_R = \sqrt{\frac{\chi + \frac{1}{d-1}}{\chi + d - 1}}.$$

Statements about velocity bounds, similar to

$$0 \leq |v_{av}| \leq \frac{L}{4G}\pi |T_R - T_L|$$

can be derived for higher dimensions.



Conclusions and Remarks

- ❖ **Universal** steady state, described by boosted black brane.
- ❖ Entanglement Entropy measures information flow.
- ❖ Mutual Information grows **monotonically** in time.
- ❖ Entanglement Entropy decrease and increase rates are **bounded**.
- ❖ Shockwaves mimic the entanglement **tsunami**.
- ❖ **Inequalities** are satisfied and violated, confirming expectations.

Universal formula:

$$f_A(\rho) \simeq 3\rho^2 - 2\rho^3$$

Outlook 1:

Callan, He, Headrick '12
Caceres, Kundu, Pedraza, Tangarife '13

➢ This bulk metric is vacuum – Null Energy Condition is satisfied.

Will time-dependent bulk spacetimes that violate NEC still satisfy the inequalities?

Outlook 2:

Bohrdt, Mendl, Endres, Knap '16

➢ The low temperature regime of a lattice model can be approximated by a CFT thermal state

Can our simple universal evolution be observed in Tensor Network calculations?

Thank you for your attention!