General Relativity and beyond

by

Mariafelicia De Laurentis

Institute for Theoretical Physics, Frankfurt

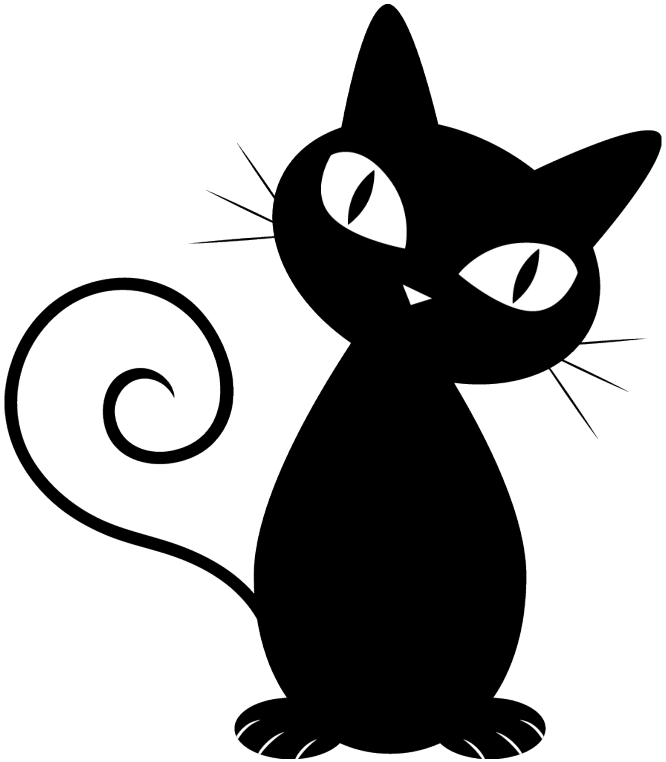
BLACK HOLE CAM

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Summary

- ✓ What we mean for a "good theory of gravity"
- ✓ Why extending General Relativity ?
- ✓ General Relativity and its shortcomings
- ✓ Extended theories of gravity
- ✓ Several examples of Extended Theories of Gravity
- ✓ Conformal transformations
- ✓ Applications :
  - Black hole solutions
  - Self gravitating systems
  - Stellar structures
  - Gravitational waves and massive modes
- ✓ Conclusions and perspectives

What we mean for a "good theory of gravity"



Requirements

1. It has to explain the astrophysical observations (e.g. the orbits of planets, self-gravitating structures)

2. it should reproduce Galactic dynamics considering the observed baryonic constituents (e.g. luminous components as stars, sub-luminous components as planets,
dust and gas) radiation and Newtonian potential which is, by assumption, extrapolated to Galactic scales

3. it should address the problem of large scale structure (e.g. clustering of galaxies) and finally cosmological dynamics

Space and time have to be entangled into a single space–time structure

The gravitational forces have to be expressed by the curvature of a metric tensor field

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \quad \text{on a four-dimensional space–time manifold}$$

The main physical object are the gravitational potentials endowed in metric coefficients (metric formulation)

On the other hand both Γ and g could be related to the gravitational quantities (metric-affine formulation)

Space–time is curved in itself and that its curvature is locally determined by the distribution of the sources (according to the former Riemann idea)

The field equations for a metric tensor $g_{\mu\nu}$, related to a given distribution of matter–energy, can be achieved by starting from the Ricci curvature scalar R which is an invariant

what is the theory that satisfy these requirements?

General Relativity

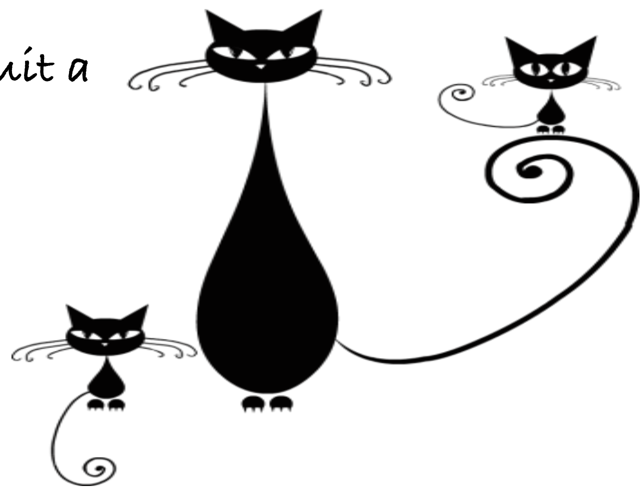
Physical and mathematical assumptions

1. The "**Principle of Relativity**", that requires all frames to be good frames for Physics, so that no preferred inertial frame should be chosen a priori (if any exist)

2. The "**Principle of Equivalence**", that amounts to require inertial effects to be locally indistinguishable from gravitational effects (in a sense, the equivalence between the inertial and the gravitational mass).

3. The "**Principle of General Covariance**", that requires field equations to be "generally covariant" (today, we would better say to be invariant under the action of the group of all space-time diffeomorphisms).

4. The causality has to be preserved (the "**Principle of Causality**", i.e. that each point of space-time should admit a universally valid notion of past, present and future).



5. the space–time structure has to be determined by either one or both of two fields, a Lorentzian metric g and a linear connection Γ

6. The metric g fixes the causal structure of space–time (the light cones) as well as its metric relations (clocks and rods);

7. The connection Γ fixes the free-fall, i.e. the locally inertial observers

8. A number of compatibility relations have to be satisfyed:

   i) photons follow null geodesics of Γ,
   ii) Γ and g can be independent, a priori, but constrained, a posteriori, by some physical restrictions (the Equivalence Principle)

9. Equivalence Principle imposes that Γ has necessarily to be the Levi-Civita connection of g

10. However if the Equivalence Principle does not holds g and Γ can be independent

Why extending General Relativity ?

General Relativity and its shortcomings

General Relativity is a theory which dynamically describes space, time and matter under the same standard

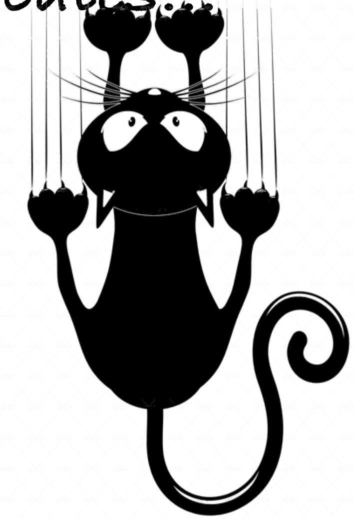
The result is a self-consistent scheme which is capable of explaining a large number of gravitational phenomena, ranging from laboratory up to cosmological scales

Despite these good results...

- GR disagrees with an increasingly number of observational data at IR-scales
- GR is not renormalizable and cannot be quantized at UV-scales

....it seems then, from ultraviolet up to infrared scales, that GR cannot be the definitive theory of Gravitation also if it successfully addresses a wide range of phenomena

Several approaches have been proposed in order to recover the validity of General Relativity at all scales...

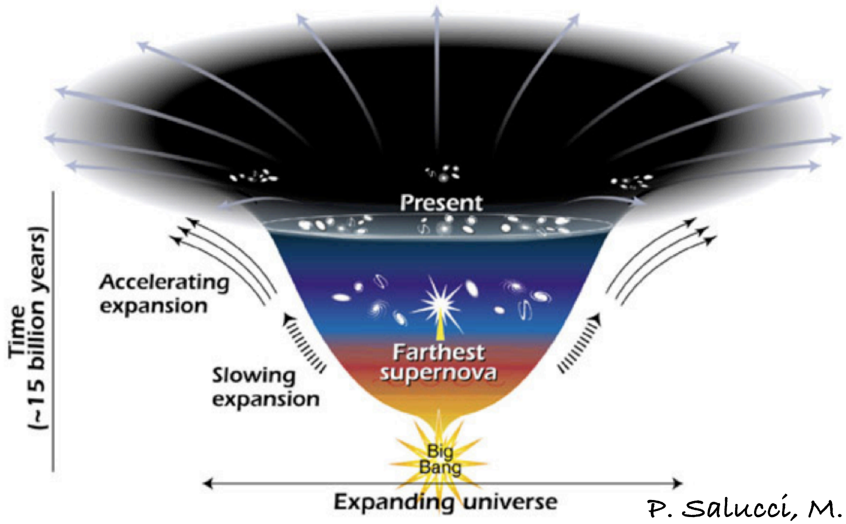


Theoretical motivations: IR scale

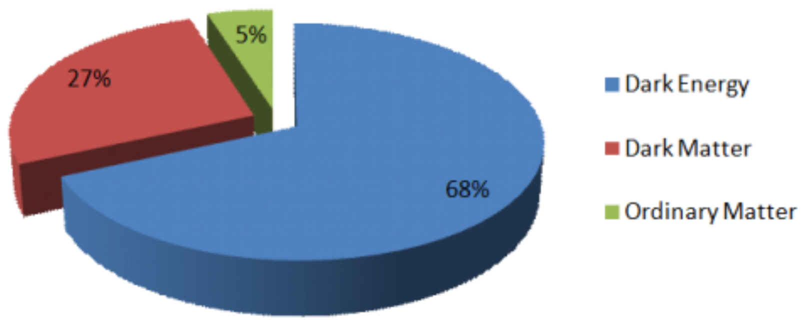
Dark Matter (DM) and Dark Energy (DE) are attempts in this way

The price of preserving the simplicity of the Hilbert Lagrangian has been the introduction of several odd behaving physical entities which, up to now, have not been revealed by any experimental fundamental scales (there are no final probe for DM and DE, e.g. at LHC)

In other words: Astrophysical observations probe the large scale effects of missing matter (DM) and the accelerating behavior of the Hubble flow (DE) but no final evidence of these ingredients exists, if we want to deal with them under the standard of quantum particles or quantum fields



NASA/A. Riess



P. Salucci, M. De Laurentis (2015): Dark matter in Galaxies. Theory, Phenomenology, experiments, to appear in The Astronomy and Astrophysics Review (2015)

The Quantum Gravity Problem: UV scales

The most important goal is to obtain an effective theory whic agrees with the other fundamental interactions at quantum level

Today, we observe and test the results of some symmetry breakings

Quantum Field theory

States of system → vectors in → Hilbert space

Fields → operators on

...a quantum mechanics framework is not consistent with gravitation

S. Capozziello, M. De Laurentis, S.D. Odintsov Eur. Phys. J. C (2012) 72:2068

.... Fields have to be quantized but $g_{\mu\nu}$ describes both of dynamical aspects of gravity and space-time background! Difficult to quantize!!!

Theoretical motivations: UV scale

To quantize the gravitational field, we have to give a quantum mechanical description of the space-time

Quantum Gravity Theory $\xrightarrow{\text{leads to}}$ unification of various interactions

Not avaible up to now!

GR assumes a classical description of matter which totally fails at subatomic scales which are the scales of the Early Universe

The situation is dark

Is General Relativity the only fundamental theory capable of explaining the gravitational interaction?

Extended Theories of Gravity

...alternative theories have been considered in order to attempt, at least, a semiclassical cheme where General Relativity and its positive results could be recovered...

the most fruitful approaches has been that of Extended Theories of Gravity which have become a sort of paradigm in the study of gravitational interaction

based on corrections and enlargements of the Einstein theory

adding higher-order curvature invariants (R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}$, $R\Box R$...) and minimally or non-minimally coupled scalar fields into dynamics ($\phi^2 R$) which come out from the effective action of quantum gravity

Y.F. Cai, S. Capozziello, M. De Laurentis, M. Saridakis, accepted in Report Progress Physics (2015)
S. Capozziello, M. De Laurentis, Phys. Rep. 509, 167 (2011)
S. Nojiri, S.D. Odintsov, Phys. Rep. 505, 59 (2011)
S. Capozziello, M. De Laurentis, V. Faraoni:, TOAJ. 2, 874 (2009).

Extended Theories of Gravity

Let us start with a general class of higher-order scalar-tensor theories in four dimensions given by the action

$$\mathcal{A}=\int d^4x\sqrt{-g}\left[F(R,\Box R,\Box^2 R,..\Box^k R,\phi)-\frac{\varepsilon}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}+\mathcal{L}_m\right]$$

In the metric approach, the field equations are obtained by varying with respect to $g_{\mu\nu}$

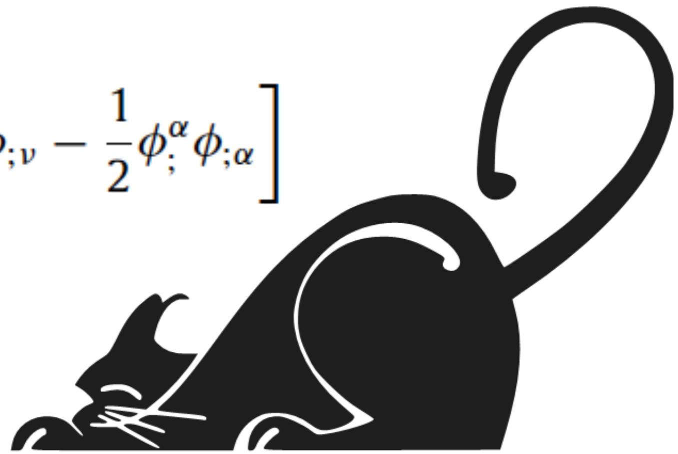
$$G^{\mu\nu}=\frac{1}{\mathcal{G}}\Bigg[\kappa T^{\mu\nu}+\frac{1}{2}g^{\mu\nu}(F-\mathcal{G}R)+(g^{\mu\lambda}g^{\nu\sigma}-g^{\mu\nu}g^{\lambda\sigma})\mathcal{G}_{;\lambda\sigma}$$

$$+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i}(g^{\mu\nu}g^{\lambda\sigma}+g^{\mu\lambda}g^{\nu\sigma})(\Box^{j-i})_{;\sigma}\left(\Box^{i-j}\frac{\partial F}{\partial\Box^i R}\right)_{;\lambda}-g^{\mu\nu}g^{\lambda\sigma}\left((\Box^{j-1}R)_{;\sigma}\Box^{i-j}\frac{\partial F}{\partial\Box^i R}\right)_{;\lambda}\Bigg]$$

where $\mathcal{G}\equiv\sum_{j=0}^{n}\Box^j\left(\frac{\partial F}{\partial\Box^j R}\right)$

The differential equations are of order (2k + 4).

The stress-energy tensor is $T_{\mu\nu}=T^{(m)}_{\mu\nu}+\frac{\epsilon}{2}\left[\phi_{;\mu}\phi_{;\nu}-\frac{1}{2}\phi^{\alpha}_{;}\phi_{;\alpha}\right]$



M. De Laurentis, MPLA 12, 1550069, (2015)

S. Capozziello, M. De Laurentis, V. Faraoni, TOAJ. 2, 874 (2009).

S. Capozziello, M. De Laurentis, Phys. Rep. 509, 167 (2011)

Extended Theories of Gravity

From the general action it is possible to obtain an interesting case by choosing
$F = F(\phi)\, R - V(\phi)$, $\varepsilon = -1$

In this case, we get $$S = \int \sqrt{-g}\left[F(\phi)R + \frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V(\phi)\right]$$

The variation with respect to $g_{\mu\nu}$ gives the second-order field equations

$$F(\phi)G_{\mu\nu} = F(\phi)\left[R_{\mu\nu} - \frac{1}{2}R_{\mu\nu}\right] = -\frac{1}{2}T^{\phi}_{\mu\nu} - g_{\mu\nu}\Box_g F(\phi) + F(\phi)_{;\mu\nu}$$

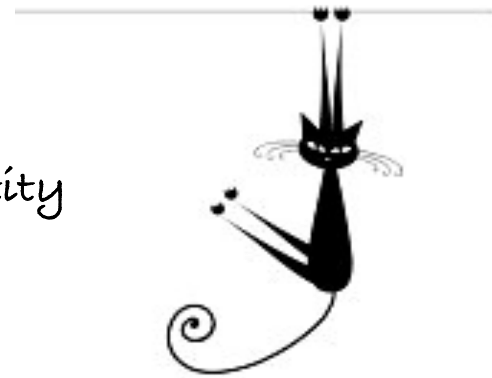
The energy-momentum tensor relative to the scalar field is

$$T^{\phi}_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{\alpha}_{;} + g_{\mu\nu}V(\phi)$$

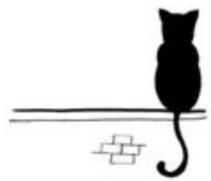
The variation with respect to φ provides the Klein–Gordon equation, i.e. the field equation for the scalar field:

$$\Box_g\phi - RF_{\phi}(\phi) + V_{\phi}(\phi) = 0$$

This last equation is equivalent to the Bianchi contracted identity



Extended Theories of Gravity



The simplest extension of GR is achieved assuming $F = f(R)$, $\varepsilon = 0$, in the action

The standard Hilbert-Einstein action is recovered for $f(R) = R$

Varying with respect to $g_{\alpha\beta}$, we get

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} = \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)$$

and, after some manipulations

$$G_{\mu\nu} = \frac{1}{f'(R)}\left\{\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) + g_{\mu\nu}\frac{[f(R) - f'(R)R]}{2}\right\}$$

where the gravitational contribution due to higher-order terms can be reinterpreted as a stress-energy tensor contribution

Considering also the standard perfect-fluid matter contribution, we have

$$G_{\alpha\beta} = \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\alpha\beta}\left[f(R) - Rf'(R)\right] + f'(R)_{;\alpha\beta} - g_{\alpha\beta}\Box f'(R)\right\} + \frac{\kappa T^{(m)}_{\alpha\beta}}{f'(R)} = \underbrace{T^{(\mathrm{curv})}_{\alpha\beta}} + \frac{T^{(m)}_{\alpha\beta}}{f'(R)}$$

In the case of GR, identically vanishes while the standard, minimal coupling is recovered for the matter contribution

is an effective stress-energy tensor constructed by the extra curvature terms

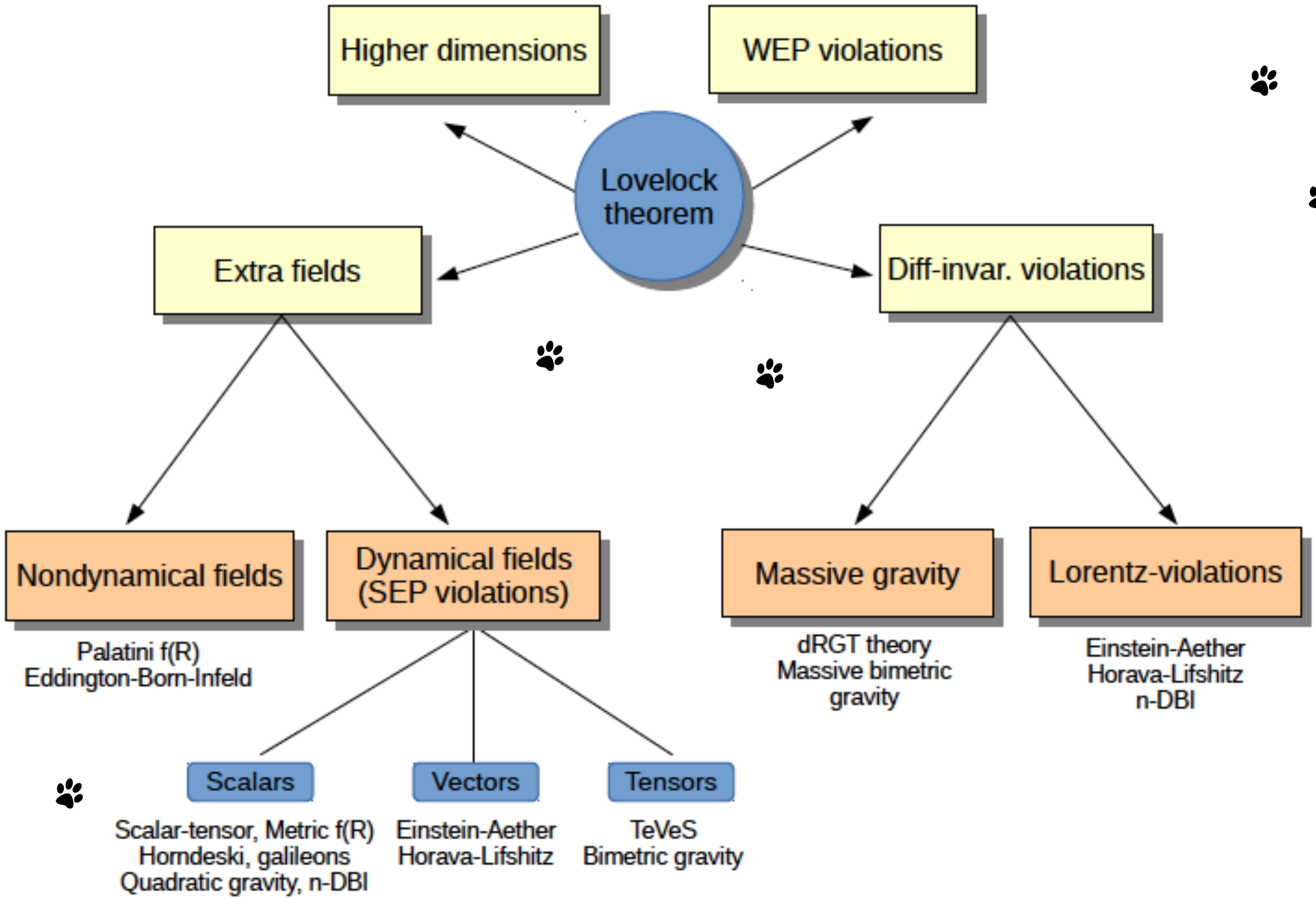
Several alternative proposals!

Is there a unification scheme to classify alternative theories?

The Lovelock theorem

In four space-time dimensions the only divergence-free symmetric rank-2 tensor constructed solely from the metric g and its derivatives up to second differential order, and preserving diffeomorphism invariance, is the Einstein tensor plus a cosmological term

In other words, some theories can be reduced to GR, other not. To this aim, a useful tool is given by the conformal transformations that we will discuss below



. Berti et al, arXiv:1501.07274 [gr-qc] (2025)

Conformal transformations

Let us now introduce conformal transformations to show that any higher-order or scalar-tensor theory, in absence of ordinary matter, e.g. a perfect fluid, is conformally equivalent to an Einstein theory plus minimally coupled scalar fields

In general, we have that, if M is a $(n+1)$- dimesional manifold and $g_{\mu\nu}$ is a metric that is assigned to it, we can generate a new metric

$$\tilde{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}$$

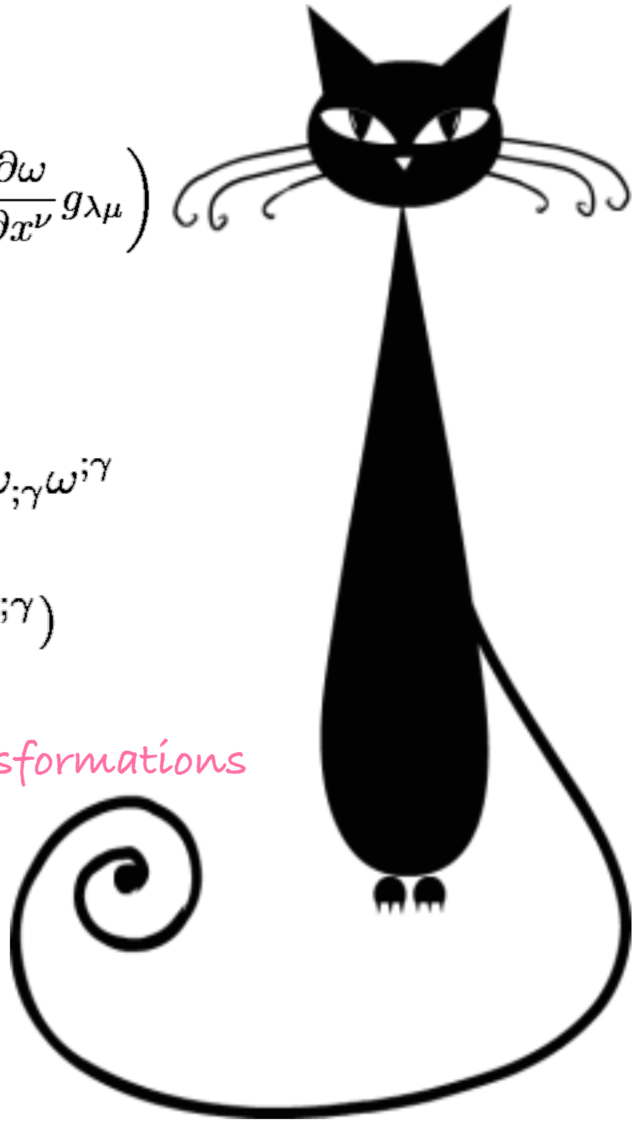
This transformation is called to conformal, since, it maintains unchanged the angles and the relations between modules of the vectors

Conformal transformations

In general, tensorial quantities are not invariant under conformal transformations, neither are the tensorial equations describing geometry and physics

In fact, the Christoffel symbols are

$$\tilde{\Gamma}^{\sigma}_{\lambda\mu} = \Gamma^{\sigma}_{\lambda\mu} + g^{\nu\sigma}\left(\frac{\partial\omega}{\partial x^{\lambda}}g_{\mu\nu} + \frac{\partial\omega}{\partial x^{\mu}}g_{\lambda\nu} - \frac{\partial\omega}{\partial x^{\nu}}g_{\lambda\mu}\right)$$



the Ricci tensor

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} - 2\omega_{;\alpha;\beta} + 2\omega_{;\alpha}\omega_{;\beta} - g_{\alpha\beta}\Box\omega - 2g_{\alpha\beta}\omega_{;\gamma}\omega^{;\gamma}$$

The Ricci scalar $\tilde{R} = e^{-2\omega}\left(R - 6\Box\omega - 6\omega_{;\gamma}\omega^{;\gamma}\right)$

The only tensor that is invariant under conformal transformations is the Weyl tensor

$$\tilde{C}^{\alpha}_{\beta\gamma\delta} = C^{\alpha}_{\beta\gamma\delta}$$

Conformal transformations

Performing the conformal transformation in f(R) field equations we get

$$\tilde{G}_{\alpha\beta} = \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\alpha\beta}\left[f(R) - Rf'(R)\right] + f'(R)_{;\alpha\beta} - g_{\alpha\beta}\Box f'(R)\right\} + 2\left(\omega_{;\alpha;\beta} + g_{\alpha\beta}\Box\omega - \omega_{;\alpha}\omega_{;\beta} + \frac{1}{2}g_{\alpha\beta}\omega_{;\gamma}\omega^{;\gamma}\right)$$

We can then choose the conformal factor to be $\omega = \frac{1}{2}\ln|f'(R)|$

Rescaling ω in such a way that **k**φ = ω, and k =√ 1/6, we obtain the Lagrangian equivalence

$$\sqrt{-g}\,f(R) = \sqrt{-\tilde{g}}\left(-\frac{1}{2}\tilde{R} + \frac{1}{2}\tilde{\phi}_{;\alpha}\tilde{\phi}^{\alpha}_{;} - \tilde{V}\right)$$

and the Einstein equations in standard form $\tilde{G}_{\alpha\beta} = \phi_{;\alpha}\phi_{;\beta} - \frac{1}{2}\tilde{g}_{\alpha\beta}\phi_{;\gamma}\phi^{;\gamma} + \tilde{g}_{\alpha\beta}V(\phi)$

with the potential

$$V(\phi) = \frac{e^{-4k\phi}}{2}\left[\mathcal{P}(\phi) - \mathcal{N}\left(e^{2k\phi}\right)e^{2k\phi}\right] = \frac{1}{2}\frac{f(R) - Rf'(R)}{f'(R)^2}.$$

Here N is the inverse function of P′(φ) and $\mathcal{P}(\phi) = \int \exp(2k\phi)\mathrm{d}\mathcal{N}$

However, the problem is completely solved if P′(φ) can be analytically inverte

In summary, a fourth-order theory is conformally equivalent to the standard second-order Einstein theory plus a scalar field

Conformal transformations

This procedure can be extended to more general theories. If the theory is assumed to be higher than fourth order, we may have Lagrangian densities of the form

$$\mathcal{L} = \mathcal{L}(R, \Box R, \ldots, \Box^k R)$$

Every $\Box$ operator introduces two further terms of derivation into the field equations.

For example a theory like

$$\mathcal{L} = R \Box R,$$

is a sixth-order theory and the above approach can be pursued by considering a conformal factor of the form

$$\omega = \frac{1}{2} \ln \left| \frac{\partial \mathcal{L}}{\partial R} + \Box \frac{\partial \mathcal{L}}{\partial \Box R} \right|$$

Conformal transformations

In general, increasing two orders of derivation in the field equations (i.e., for every term $\Box R$), corresponds to adding a scalar field in the conformally transformed frame

A sixth-order theory can be reduced to an Einstein theory with two minimally coupled scalar fields; a $2n$-order theory can be, in principle, reduced to an Einstein theory plus $(n-1)$-scalar fields

S. Gottlober, H-J Schmidt, and A A Starobinsky, Class. Quantum Grav. 7, 893 (1990)

Conformal transformations work at three levels:

(i) on the Lagrangian of the given theory;
(ii) on the field equations;
(iii) on the solutions.

They allow to classify gravitational degrees of freedom and reduce any higher-order theory to Einstein plus scalar field

The Palatini formalism

The Palatini formalism (metric-affine formulation) comes out in the case in which g and Γ are two independent object . Equivalence Principle could not hold any more

Let usconsider an $f(\mathcal{R}) \Rightarrow$ $\mathcal{R} \equiv \mathcal{R}(g, \Gamma) \equiv g^{\alpha\beta}\mathcal{R}_{\alpha\beta}(\Gamma)$

The field equations derived with the Palatini variational principle are

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)}(\Gamma) - \frac{f(\mathcal{R})}{2} g_{\mu\nu} = T^{(m)}_{\mu\nu},$$

$$\nabla^{\Gamma}_{\alpha}\left[\sqrt{-g}\, f'(\mathcal{R}) g^{\mu\nu}\right] = 0,$$

S.Capozziello, M.F.De Laurentis, L.Fatibene, M.Ferraris, S.Garruto arXiv 1509.08008 (2015)

S. Capozziello, M. De Laurentis, M. Francaviglia, S. Mercadante: Foundations of Physics 39, 1161 (2009)

is a symmetric tensor density of weight 1, which naturally leads to the introduction of a new metric $h_{\mu\nu}$ conformally related to $g_{\mu\nu}$

$$\sqrt{-g}\, f'(\mathcal{R})\, g^{\mu\nu} = \sqrt{-h}\; h^{\mu\nu}$$

With this definition $\Gamma^{\alpha}_{\mu\nu}$ is the Levi-Civita connection of the metric $h_{\mu\nu}$, with the only restriction that the conformal factor relating $g_{\mu\nu}$ and $h_{\mu\nu}$ be non-degenerate

In the case of the Hilbert-Einstein Lagrangian it is $f'(R) = 1$

The Palatini formalism

The conformal transformation $g_{\mu\nu} \longrightarrow h_{\mu\nu} = f'(\mathcal{R})\, g_{\mu\nu}$ implies $\mathcal{R}_{(\mu\nu)}(\Gamma) = \mathcal{R}_{\mu\nu}(h)$

It is useful to consider the trace of the field equation

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = g^{\alpha\beta}T^{(m)}_{\alpha\beta} \equiv T^{(m)}$$

We refer to this scalar equation as the structural equation of space-time

In vacuo and in the presence of conformally invariant matter with $T^{(m)} = 0$, this scalar equation admits constant solutions

In these cases, Palatini $f(R)$-gravity reduces to GR with a cosmological constant

In the case of interaction with matter fields, the structural equation, if explicitly solvable, provides in principle an expression $R = F(T^{(m)})$ and, as a result, both $f(R)$ and $f'(R)$ can be expressed in terms of $T^{(m)}$.

This fact allows one to express, at least formally, R in terms of $T^{(m)}$, which has deep consequences for the description of physical systems

Matter rules the bi-metric structure of space-time and, consequently, both the geodesic and metric structures which are intrinsically different

The Palatini formalism to non-minimally coupled scalar-tensor theories

The scalar-tensor action can be generalized as

$$S_1 = \int d^4x\sqrt{-g}\left[F(\phi)\mathcal{R} - \frac{\epsilon}{2}\overset{g}{\nabla}_\mu\phi\overset{g}{\nabla}{}^\mu\phi - V(\phi) + \mathcal{L}^{(m)}\left(\Psi, \overset{g}{\nabla}\Psi\right)\right]$$

The field equations for the metric $g_{\mu\nu}$ and the connection $\Gamma^\alpha{}_{\mu\nu}$ are

$$F(\phi)\left(\mathcal{R}_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}\mathcal{R}\right) = T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu},$$

$$\nabla^\Gamma_\alpha\left[\sqrt{-g}\,F(\phi)g^{\mu\nu}\right] = 0,$$

The equation of motion of the matter fields is

$$\epsilon\Box\phi = V_\phi(\phi) + F_\phi(\phi)\mathcal{R},$$

$$\frac{\delta\mathcal{L}^{(m)}}{\delta\Psi} = 0.$$

the structural equation of space-time implies that

$$\mathcal{R} = -\frac{\left(T^{(\phi)} + T^{(m)}\right)}{F(\phi)}$$

where we must require that $F(\varphi) > 0$

The bi-metric structure of space-time is thus defined by the ansatz

$$\sqrt{-g}\;F(\phi)g^{\mu\nu} = \sqrt{-h}\;h^{\mu\nu}$$

so that

$$h_{\mu\nu} = F(\phi)\,g_{\mu\nu}$$

It follows that in vacuo $T^{(\varphi)} = 0$ and $T^{(m)} = 0$ this theory is equivalent to vacuum GR

If $F(\varphi) = F_0 =$ const. we recover GR with a minimally coupled scalar field

Equivalence between scalar–tensor and metric $f(R)$-gravity

(a realization of Lovelock approach)

In metric $f(R)$-gravity, we introduce the scalar $\varphi \equiv R$; then the action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\, f(R) + S^{(m)}$$

is rewritten in the form

coincides with if $\varphi = R$.

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\, [\psi(\phi)R - V(\phi)] + S^{(m)}$$

when $f''(R) \neq 0$, where $\psi = f'(\phi), \qquad V(\phi) = \phi f'(\phi) - f(\phi)$

Vice-versa, let us vary the action with respect to φ, which leads to $R\dfrac{d\psi}{d\phi} - \dfrac{dV}{d\phi} = (R-\phi)f''(R) = 0.$

The action has the Brans–Dicke form

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\psi R - \frac{\omega}{2}\nabla^\mu\psi\nabla_\mu\psi - U(\psi)\right] + S^{(m)}$$

with Brans–Dicke field ψ, Brans–Dicke parameter $\omega = 0$, and potential $U(\psi) = V[\varphi(\psi)]$

An $\omega = 0$ Brans–Dicke theory was originally studied for the purpose of obtaining a Yukawa correction to the Newtonian potential in the weak-field limit and called "O'Hanlon theory" or "massive dilaton gravity"

The variation of the action yields the field equations

$$G_{\mu\nu} = \frac{\kappa}{\psi}T^{(m)}_{\mu\nu} - \frac{1}{2\psi}U(\psi)g_{\mu\nu} + \frac{1}{\psi}\left(\nabla_\mu\nabla_\nu\psi - g_{\mu\nu}\Box\psi\right)$$

$$3\Box\psi + 2U(\psi) - \psi\frac{dU}{d\psi} = \kappa T^{(m)}.$$

Equivalence between scalar–tensor and Palatini $f\,(R)$-gravity

The Palatini action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,f(\mathcal{R}) + S^{(m)}$$

is equivalent to

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,[f(\chi) + f'(\chi)(\mathcal{R}-\chi)] + S^{(m)}$$

It is straightforward to see that the variation of this action with respect to χ yields $\chi = \mathcal{R}$

We can now use the field $\varphi \equiv f\,'(\chi)$ and the fact that the curvature $\mathcal{R}$ is the (metric) Ricci curvature of the new metric $h_{\mu\nu} = f\,'(\mathcal{R})\, g_{\mu\nu}$ conformally related to $g_{\mu\nu}$

Using now the well known transformation property of the Ricci scalar under conformal rescalings

$$\mathcal{R} = R + \frac{3}{2\phi}\nabla^\alpha\phi\nabla_\alpha\phi - \frac{3}{2}\Box\phi$$

and discarding a boundary term, the action can be presented in the form

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\phi R + \frac{3}{2\phi}\nabla^\alpha\phi\nabla_\alpha\phi - V(\phi)\right] + S^{(m)},$$

where $V(\phi) = \phi\chi(\phi) - f\,[\chi(\phi)]$

This action is clearly that of a Brans–Dicke theory with Brans–Dicke parameter $\omega = -3/2$ and a potential

A. Borowiec, S. Capozziello, M. De Laurentis, F. S. N. Lobo, A. Paliathanasis, M. Paolella, A. Wojnar, PR D 91, 2, 023517 (2015)

The interpretation of conformal frames

The conformal transformation from the Jordan to the Einstein frame is a mathematical map which allows one to study several aspects any Extended Theories of Gravity

having now available both the Jordan and the Einstein conformal frames, one wonders whether the two frames are also physically equivalent or only mathematically related

the problem is whether the physical meaning of the theory is "preserved" or not by the use of conformal transformations

One has now the metric

$g_{\mu\nu}$ and its conformal cousin $\breve{g}_{\mu\nu}$

and the question has been posed of which one is the "physical metric", i.e., the metric from which curvature, geometry, and physical effects should be calculated and compared with experiment

The interpretation of conformal frames

The question of Jordan frame and Einstein frame can be summarized according to the fact that

- geometry can be modified (left hand side of Einstein equations) i.e. the Jordan frame or
- the source can be modified preserving the Einstein tensor (right hand side Einstein equations), i.e. the Einstein frame.

This means that matter remains minimally coupled in the Jordan frame while it is non-minimally coupled in the Einstein frame

From a genuine physical point of view the Jordan frame is the physical frame, since matter traces the geodesic structure

EoS	$\longleftrightarrow$	$\mathcal{L}_{ST}$	$\longleftrightarrow$	$\mathcal{L}_{f(R)}$
$\updownarrow$		$\updownarrow$		$\updownarrow$
Einstein eqs.	$\longleftrightarrow$	ST eqs.	$\longleftrightarrow$	$f(R)$ eqs.
$\updownarrow$		$\updownarrow$		$\updownarrow$
E frame sol.	$\longleftrightarrow$	E frame sol. $+\phi$	$\longleftrightarrow$	J frame sol.

Applications to astrophysics

- ✓ Are needed to probe Extended Theories of Gravity
- ✓ Could be a signature at IR-scales
- ✓ Could address phenomena out of GR
- ✓ Could probe Dark Matter and Dark Energy effects

Some exact Black hole solutions

Let us consider an analytic function f(R), the variational principle for this action is

$$\delta \int d^4x\sqrt{-g}[f(R) + \mathcal{X}\mathcal{L}_m] = 0,$$

By varying with respect to the metric, we obtain the field equations

$$\begin{cases} H_{\mu\nu} = f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\Box f'(R) = \mathcal{X}T_{\mu\nu} \\ H = g^{\rho\sigma}H_{\rho\sigma} = 3\Box f'(R) + f'(R)R - 2f(R) = \mathcal{X}T, \end{cases}$$

The most general spherically symmetric solution ca be written as follows:

$$ds^2 = m_1(t', r')\,dt'^2 + m_2(t', r')\,dr'^2 + m_3(t', r')\,dt'\,dr' + m_4(t', r')\,d\Omega,$$

We can consider a coordinate transformation that maps metric in a new one where the off-diagonal term vanishes and $m_4(t', r') = -r^2$, that is,

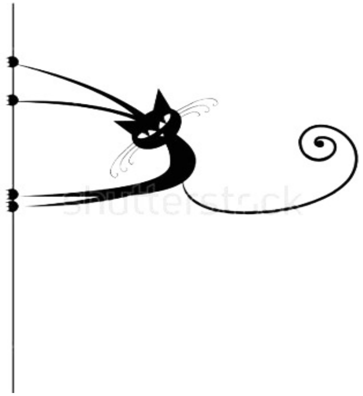
$$ds^2 = g_{tt}(t, r)\,dt^2 - g_{rr}(t, r)\,dr^2 - r^2\,d\Omega.$$

Spherical symmetric solution

...by inserting this metric into the field equations , one obtains

$$\begin{cases} f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + \mathcal{H}_{\mu\nu} = \mathcal{X}T_{\mu\nu} \\ f'(R)R - 2f(R) + \mathcal{H} = \mathcal{X}T, \end{cases}$$

...where the two quantities $H_{\mu\nu}$ and H read



$$\begin{aligned}\mathcal{H}_{\mu\nu} = -f''(R)\Big\{R_{,\mu\nu} - \Gamma^{t}_{\mu\nu}R_{,t} - \Gamma^{r}_{\mu\nu}R_{,r} - g_{\mu\nu}\big[\big(g^{tt}{}_{,t} + g^{tt}(\ln\sqrt{-g})_{,t}\big)R_{,t} \\ + \big(g^{rr}{}_{,r} + g^{rr}(\ln\sqrt{-g})_{,r}\big)R_{,r} + g^{tt}R_{,tt} + g^{rr}R_{,rr}\big]\Big\} \\ - f'''(R)\big[R_{,\mu}R_{,\nu} - g_{\mu\nu}\big(g^{tt}R_{,t}{}^{2} + g^{rr}R_{,r}{}^{2}\big)\big]\end{aligned}$$

$$\begin{aligned}\mathcal{H} = g^{\sigma\tau}\mathcal{H}_{\sigma\tau} = 3f''(R)\big[\big(g^{tt}{}_{,t} + g^{tt}(\ln\sqrt{-g})_{,t}\big)R_{,t} + \big(g^{rr}{}_{,r} + g^{rr}(\ln\sqrt{-g})_{,r}\big)R_{,r} \\ + g^{tt}R_{,tt} + g^{rr}R_{,rr}\big] + 3f'''(R)\big[g^{tt}R_{,t}{}^{2} + g^{rr}R_{,r}{}^{2}\big].\end{aligned}$$

After some calculations we can find out general solutions for the field equations giving the dependence of the Ricci scalar on the radial coordinate r

$$\mathrm{d}s^2 = (\alpha + \beta r)\,\mathrm{d}t^2 - \frac{1}{2}\frac{\beta r}{\alpha + \beta r}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega$$

The same procedure can be worked out with Noether symmetries approach.

M. De Laurentis, M. Paolella, S. Capozziello, PRD 91, 083531 (2015)

M. De Laurentis, L. Sebastiani, submitted to PRD (2015)

S. Capozziello, M. De Laurentis, A. Stabile, Class. Quantum Grav. 27, 165008, (2010)

Axially symmetry from spherical symmetry

It is possible to obtain an axially symmetric solution starting from spherical symmetry using the tedrad fromalism

$$g^{\mu\nu} = \begin{pmatrix} 0 & \sqrt{\frac{2}{\beta r}} & 0 & 0 \\ \cdot & -2-\frac{2\alpha}{\beta r} & 0 & 0 \\ \cdot & \cdot & -1/r^2 & 0 \\ \cdot & \cdot & \cdot & -1/(r^2\sin^2\theta) \end{pmatrix} \Longleftrightarrow g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu$$

M. De Laurentis, R. Giambò submitted to CQG (2015)

The complex tetrad null vectors are

$$\begin{cases} l^\mu = \delta^\mu_1 \\ n^\mu = -\left[1+\frac{\alpha}{\beta}\left(\frac{1}{\bar{r}}+\frac{1}{r}\right)\right]\delta^\mu_1 + \sqrt{\frac{2}{\beta}}\frac{1}{\sqrt[4]{\bar{r}r}}\delta^\mu_0 \\ m^\mu = \frac{1}{\sqrt{2}\bar{r}}\left(\delta^\mu_2 + \frac{i}{\sin\theta}\delta^\mu_3\right). \end{cases}$$

The new metric is

$$g_{\mu\nu} = \begin{pmatrix} \frac{r(\alpha+\beta r)+a^2\beta\cos^2\theta}{\Sigma} & 0 & 0 & \frac{a(-2\alpha r-2\beta\Sigma^2+\sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{2\Sigma} \\ \cdot & -\frac{\beta\Sigma^2}{2\alpha r+\beta(a^2+r^2+\Sigma^2)} & 0 & 0 \\ \cdot & \cdot & -\Sigma^2 & 0 \\ \cdot & \cdot & \cdot & -\left[\Sigma^2-\frac{a^2(\alpha r+\beta\Sigma^2-\sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{\Sigma}\right]\sin^2\theta \end{pmatrix}.$$

S. Capozziello, M. De Laurentis, A. Stabile, Class. Quantum Grav. 27, 165008, (2010)
M. De Laurentis, EPJC 71, 1675, (2011)

Dynamics of a particle around a black hole

Standard Hamiltonian formalism for geodesic motion

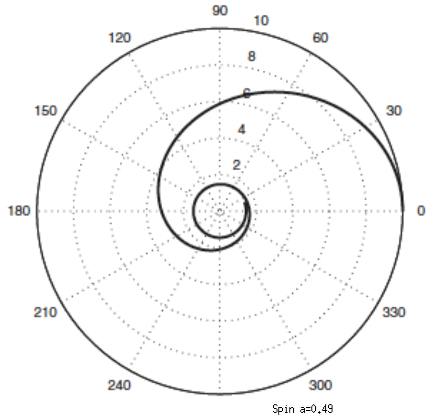
The Hamiltonian reads:

$$H = -p_0 = \left[\frac{p_i g^{0i}}{g^{00}} + \left[\left(\frac{p_i g^{0i}}{g^{00}} \right)^2 - \frac{m^2 + p_i p_j g^{ij}}{g^{00}} \right]^{1/2} \right]$$

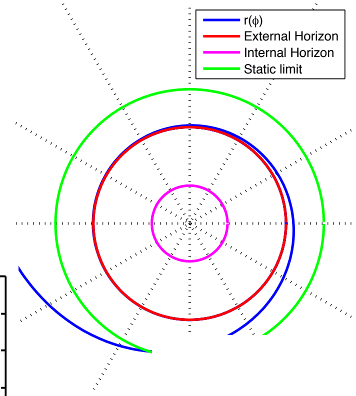
with the equations of motion $\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}$ and $\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x^i}$

Solution of Hamilton's equations

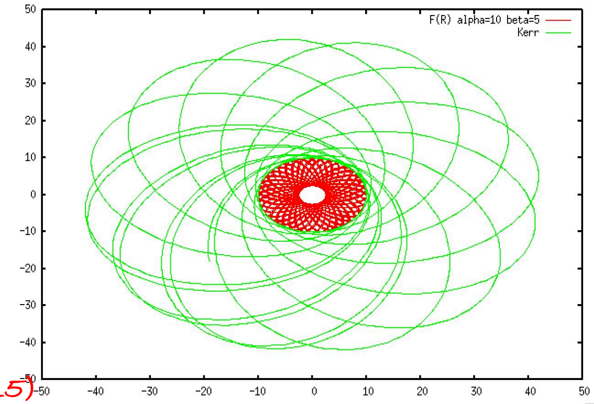
Specifies the initial value of the vector in the phase space: position and momenta



that gives the orbits, the horizon and



Free evolution and use of Carter's constant as a check of the accuracy of the numerical integration



Comparing orbits with GR (Kerr solution)

M. De Laurentis et al. in preparation

F. Tamburini, M. De Laurentis, R. Kerr accepted in PRL (2015)

M. De Laurentis, S. Capozziello, Nova publisher ISBN: 978-1-61942-929-1 (2012)

GW emission from a black holes

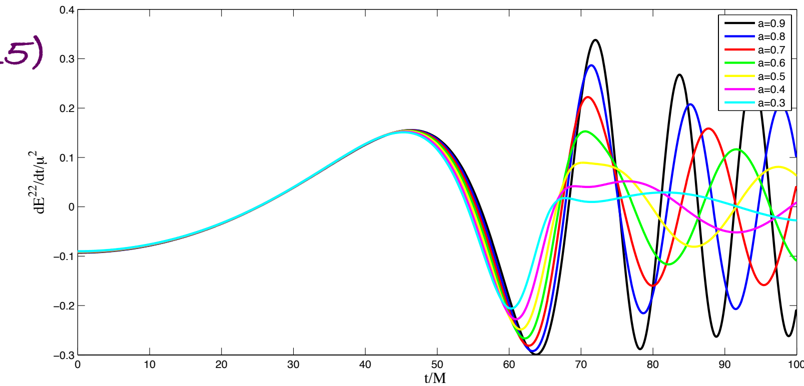
One would need to use a consistent perturbation treatment: time-domain solution of modified Zerilli-Regge-Wheeler equation

M. De Laurentis, in preparation (2015)

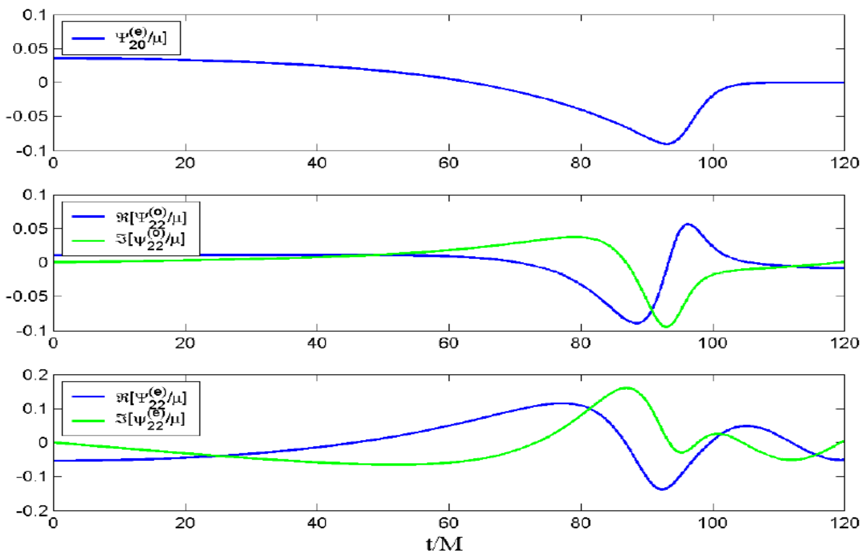
M. De Laurentis, A. Spallicci submitted to CQG (2015)

Use the multipole expansion of gravitational radiation to gain an idea about the general qualitative features of the GWs

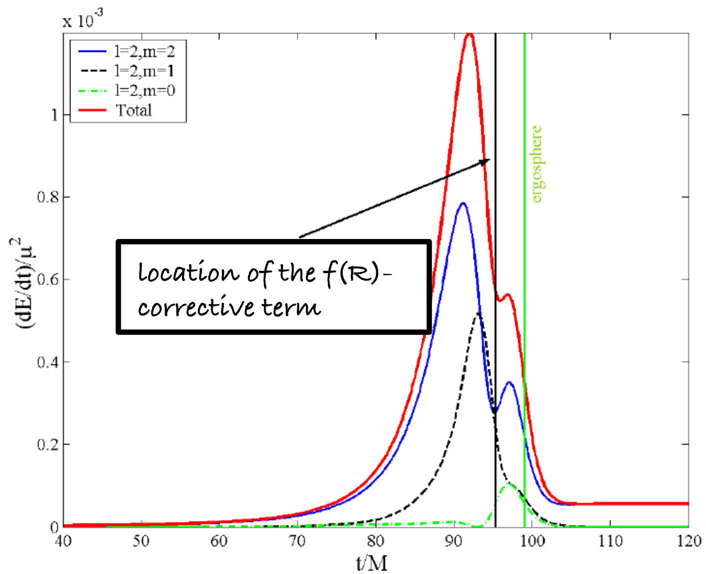
Instantaneous angular momentum loss



Precursor + burst structure of waveforms



GW-luminosity



Hydrostatic equilibrium and Stellar structures

Field equations at O (2)-order, that is at the Newtonian level, are

$$R_{tt}^{(2)} - \frac{R^{(2)}}{2} - f''(0) \triangle R^{(2)} = \mathcal{X} T_{tt}^{(0)}$$

$f^n(R) = f^n(R^{(2)} + O(4)) = f^n(0) + f^{n+1}(0) R^{(2)} + \ldots$

$$-3f''(0) \triangle R^{(2)} - R^{(2)} = \mathcal{X} T^{(0)},$$

We recall that the energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu - p g_{\mu\nu};$$

Being the pressure contribution negligible in the field equations in the Newtonian approximation, we have

$$\triangle \Phi + \frac{R^{(2)}}{2} + f''(0) \triangle R^{(2)} = -\mathcal{X}\rho$$

modified Poisson equation

$$3f''(0) \triangle R^{(2)} + R^{(2)} = -\mathcal{X}\rho,$$

S. Capozziello, M. De Laurentis Ann. Phys. 524, 545 (2012)

For $f''(R) = 0$ we have the standard Poisson equation $\triangle\Phi = -4\pi G\rho$

From the Bianchi identity we have $T^{\mu\nu}{}_{;\mu} = 0 \rightarrow \dfrac{\partial p}{\partial x^k} = -\dfrac{1}{2}(p + \epsilon)\dfrac{\partial \ln g_{tt}}{\partial x^k}.$

I. De Martino M. De Laurentis, F. Atrio-Barandela, S. Capozziello, MNRAS 442, 921 (2014).

Hydrostatic equilibrium

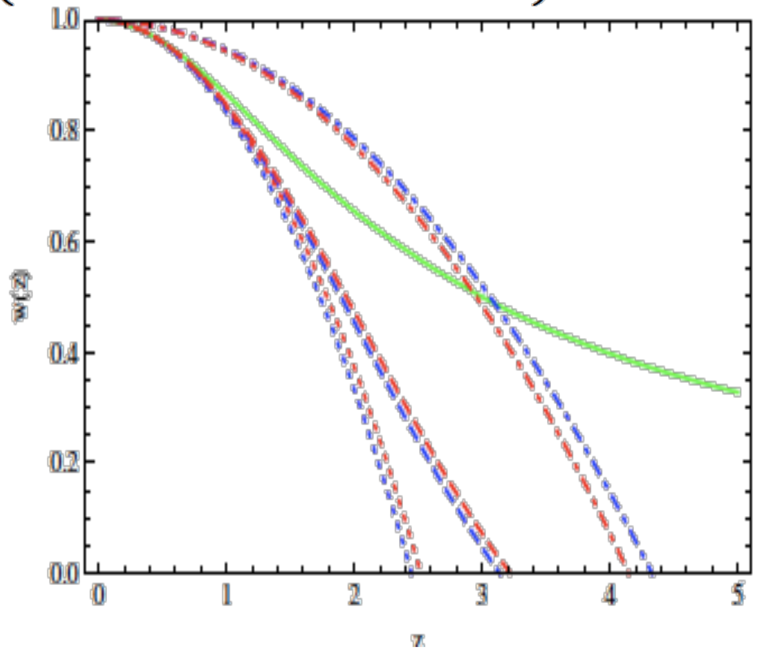
Let us suppose that matter still
satisfies a polytropic equation $p = \kappa^{\gamma}\rho^{\gamma}$

we obtain an integro-differential equation for the gravitational potential , that is

$$\frac{d^2w(z)}{dz^2} + \frac{2}{z}\frac{dw(z)}{dz} + w(z)^n = \frac{m\xi_0}{8}\frac{1}{z}\int_0^{\xi/\xi_0} dz'\, z' \left\{ e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|} \right\} w(z')^n$$

Lané-Emden equation in f(R)-gravity

We find the radial profiles of the
gravitational potential by solving for some
values of n (polytropic index)



New solutions are physically relevant and
could explain exotic systems out of Main
Sequence (magnetars, variable stars).

R. Farinelli, M. De Laurentis, S. Capozziello, S.D. Odintsov, MNRAS 440, 3, 2894. (2014)
S. Capozziello, M. De Laurentis, A. Stabile, S.D. Odintsov, PRD 83, 064004, (2011)

Self gravitating systems

Field equations in f(R)-gravity give rise to the Modified Poisson equations.
We know that

$$R^{(2)} \simeq \frac{1}{2}\nabla^2 g^{(2)}_{00} - \frac{1}{2}\nabla^2 g^{(2)}_{ii}$$

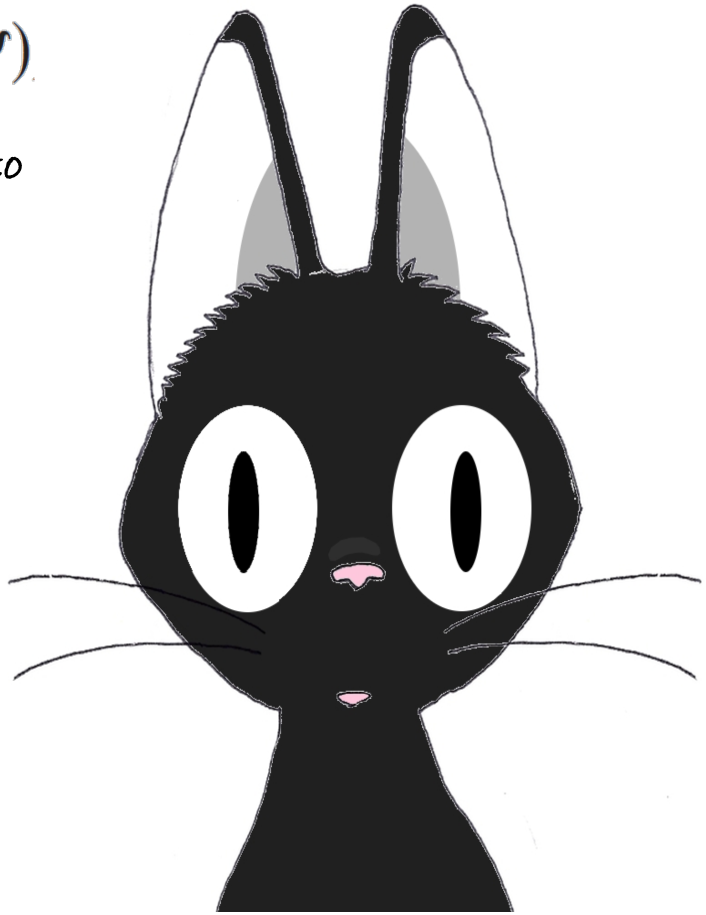
Also we well known that $R^{(2)} \simeq \nabla^2(\Phi - \Psi)$

✓ Ψ is the further gravitational potential related to the metric component $g^{(2)}_{ii}$

...and then the field equations assume this form

$$\nabla^2\Phi + \nabla^2\Psi - 2f''(0)\nabla^4\Phi + 2f''(0)\nabla^4\Psi = 2\mathcal{X}\rho$$

$$\nabla^2\Phi - \nabla^2\Psi + 3f''(0)\nabla^4\Phi - 3f''(0)\nabla^4\Psi = -\mathcal{X}\rho.$$



S. Capozziello, M. De Laurentis Ann. Phys. 524, 545 (2012)

Jeans instability in f(R)-gravity

Dynamics and collapse of collisionless self-gravitating systems is described by the coupled collisionless Boltzmann and Poisson equations

$$-i\omega f_1 + \vec{v}\cdot\left(\vec{ik}f_1\right) - \left(\vec{ik}\Phi_1\right)\cdot\frac{\partial f_0}{\partial\vec{v}} = 0,$$

$$-k^2(\Phi_1+\Psi_1) - 2\alpha k^4(\Phi_1-\Psi_1) = 16\pi G\int f_1 d\vec{v},$$

$$k^2(\Phi_1-\Psi_1) - 3\alpha k^4(\Phi_1-\Psi_1) = 8\pi G\int f_1 d\vec{v}.$$

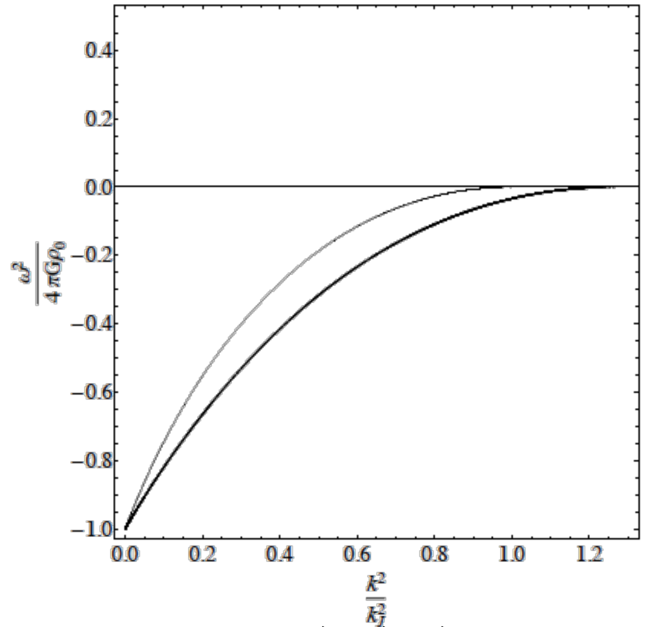
Combining the above equations we obtain a relation between Φ_1 and Ψ_1

$$\Psi_1 = \frac{3-4\alpha k^2}{1-4\alpha k^2}\Phi_1$$

A dispersion equation is achieved for neutral dust-particle systems where a generalized Jeans wave number is obtained

$$\frac{3k^4}{k_j^4} + \frac{k^2}{k_j^2} = \left(\frac{4k^2}{k_j^2}+1\right)[1-\sqrt{\pi}xe^{x^2}(1-\mathrm{erf}[x])] = 0.$$

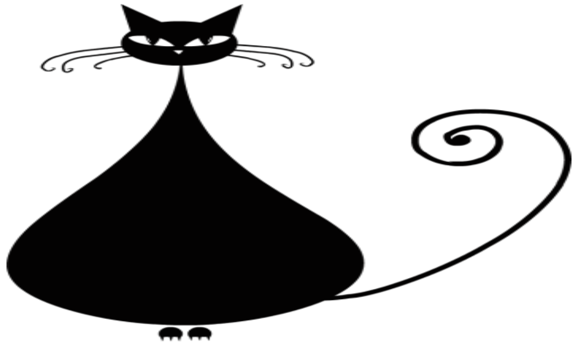
Plot of dispersion function



The bold line indicates the plot of modified dispersion relation while the thin line indicates the plot of the standard dispersion

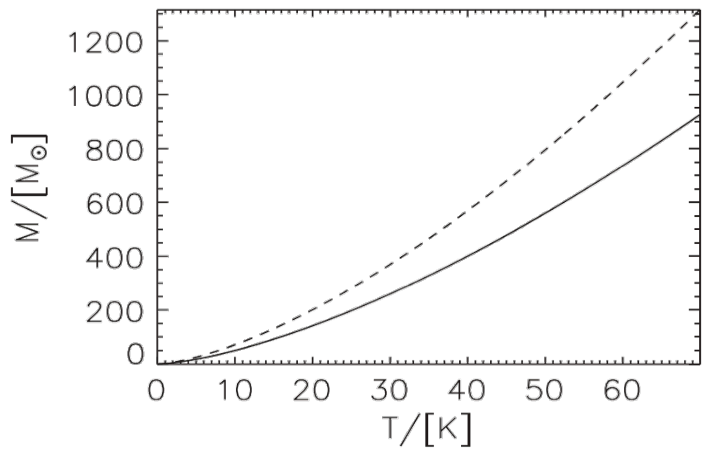
S. Capozziello, M. De Laurentis, I. De Martino, M. Formisano, S.D. Odintsov PRD 85, 044022, (2011)

The Jeans mass limit in f(R)-gravity



We have also compared the behavior with the temperature of the Jeans mass for various types of interstellar molecular clouds

$$\tilde{M}_J = 6\sqrt{\frac{6}{(3+\sqrt{21})^3}}M_J$$



In our model the limit (in unit of mass) to start the collapse of an interstellar cloud is lower than the classical one advantaging the structure formation.

M. De Laurentis, S. Capozziello, Nova publisher ISBN: 978-1-61942-929-1 (2012)

Subject	T (K)	n (10^8 m^{-3})	μ	M_J ($M_\odot$)	$\tilde{M}_J$ ($M_\odot$)
Diffuse hydrogen clouds	50	5.0	1	795.13	559.68
Diffuse molecular clouds	30	50	2	82.63	58.16
Giant molecular clouds	15	1.0	2	206.58	145.41
Bok globules	10	100	2	11.24	7.91

Massive and massless modes

We have linearized the field equations for higher order theories that contain scalar invariants other than the Ricci scalar

$$S=\int d^4x\sqrt{-g}f(R,P,Q)$$

where

$$P\equiv R_{ab}R^{ab},$$
$$Q\equiv R_{abcd}R^{abcd}$$

Varying with respect to the metric, one gets the field equations

$$\begin{aligned}FG_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}\left(f-R\,F\right)-\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)F\\ &-2\left(f_P R^a_\mu R_{a\nu}+f_Q\,R_{abc\mu}R^{abc}{}_\nu\right)\\ &-g_{\mu\nu}\nabla_a\nabla_b(f_P R^{ab})-\Box(f_P R_{\mu\nu})\\ &+2\nabla_a\nabla_b\left(f_P\;R^a{}_{(\mu}\delta^b{}_{\nu)}+2f_Q\;R^a{}_{(\mu\nu)}{}^b\right)\end{aligned}$$

To find the various GW modes, we need to linearize gravity around a Minkowski background:

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$$
$$\Phi=\Phi_0+\delta\Phi$$

Perturbing the field equations, ... we get

$$\frac{1}{2}\left(k^2-k^4\frac{f_{P0}+4f_{Q0}}{F_0}\right)\bar{h}_{\mu\nu}=(\eta_{\mu\nu}k^2-k_\mu k_\nu)\frac{\delta\Phi}{F_0}+(\eta_{\mu\nu}k^2-k_\mu k_\nu)h_f$$

$$h_f\equiv-\frac{\delta\Phi}{F_0}$$

The equation for the perturbations is

$$\left(k^2+\frac{k^4}{m^2_{spin2}}\right)\bar{h}_{\mu\nu}=0$$

$$m^2_{spin2}\equiv-\frac{F_0}{f_{P0}+4f_{Q0}}$$

We have a modified dispersion relation which corresponds to a massless spin-2 field ($k^2=0$) and massive 2-spin ghost mode

$$k^2=\frac{F_0}{\frac{1}{2}f_{P0}+2f_{Q0}}\equiv-m^2_{spin2}$$

S. Capozziello, G. Basini, M. De Laurentis, Eur. Phys. J. C 71, 1679 (2011)
S.Capozziello, C. Corda, M. De Laurentis: PLB 669, 255 (2008)

Massive and massless modes

$$\left(k^2 + \frac{k^4}{m^2_{spin2}}\right)\bar{h}_{\mu\nu} = 0$$

Solutions are plane waves

$$\bar{h}_{\mu\nu} = A_{\mu\nu}(\overrightarrow{p}) \cdot exp(ik^\alpha x_\alpha) + cc$$

$$\Box h_f = m_s^2 h_f$$

Massive mode

$$h_f = a(\overrightarrow{p}) \cdot exp(iq^\alpha x_\alpha) + cc$$

For $k^2=0$ mode → a massless spin-2 field with two independent polarizations plus a scalar mode

For $k^2 \neq 0$ mode → a massive spin-2 ghost mode and there are five independent polarization tensors plus a scalar mode

In the z direction, a gauge in which only A_{11}, A_{22}, and $A_{12} = A_{21}$ are different to zero can be chosen. The condition $h = 0$ gives $A_{11} = -A_{22}$.

In this frame we may take the bases of polarizations defined in this way

$$e^{(+)}_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e^{(\times)}_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e^{(s)}_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

...the characteristic amplitude

$$h_{\mu\nu}(t,z) = \underbrace{A^+(t-z)e^{(+)}_{\mu\nu} + A^\times(t-z)e^{(\times)}_{\mu\nu}}_{\text{two standard polarizations of GW arise from GR}} + \underbrace{h_s(t - v_G z)e^s_{\mu\nu}}_{\text{the massive field arising from the generic high-order theory}}$$

Classification of gravitational modes

When the spin-2 field is massive, we have six polarizations defined by

$$e^{(+)}_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e^{(\times)}_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e^{(B)}_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad e^{(C)}_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$e^{(D)}_{\mu\nu} = \frac{\sqrt{2}}{3}\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad e^{(s)}_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

....and the amplitude in terms of the 6 polarization states as

$$h_{\mu\nu}(t,z) = A^{+}(t - v_{G_{s2}}z)e^{(+)}_{\mu\nu} + A^{\times}(t - v_{G_{s2}}z)e^{(\times)}_{\mu\nu} + B^{B}(t - v_{G_{s2}}z)e^{(B)}_{\mu\nu}$$
$$+ C^{C}(t - v_{G_{s2}}z)e^{(C)}_{\mu\nu} + D^{D}(t - v_{G_{s2}}z)e^{(D)}_{\mu\nu} + h_s(t - v_G z)e^{s}_{\mu\nu}.$$

is the group velocity of the massive spin-2 field and is given by

$$v_{G_{s2}} = \frac{\sqrt{\omega^2 - m_{s2}^2}}{\omega}$$

K. Bamba, S. Capozziello M. De Laurentis, S. Nojiri, D. Saez-Gomez PLB 727, 194 (2013)
M. De Laurentis, S. Capozziello, G. Basini MPLA A 24, 0217(2012)
C. Bogdanos, S.Capozziello, M. De Laurentis, S. Nesseris, Astrpart. Phys. 34 (2010) 236

Classification of gravitational modes

The fact that 6 polarization states emerge is in agreement with the possible allowed polarizations of spin-2 field

H. van Dam and M. J. G. Veltman, Nucl. Phys. B 2 ,397 (1970).

In fact the spin degenerations is

$$d = (2s+1) \qquad m_g \neq 0 \quad \Longrightarrow \quad s = 2,\ d = 5$$

$$d = 2s \qquad m_g = 0 \quad \Longrightarrow \quad s = 1,\ d = 2$$

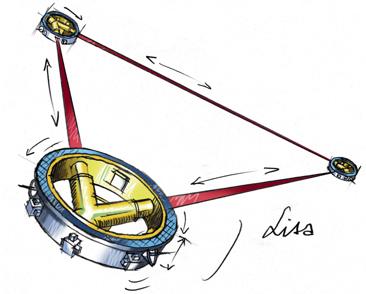
$$d = (2s+1) \qquad m_g \neq 0 \quad \Longrightarrow \quad s = 0,\ d = 1$$

An interesting fact is this result is perfectly in agreement with the fundamental Riemann theorem stating that in a N –dimensional space,

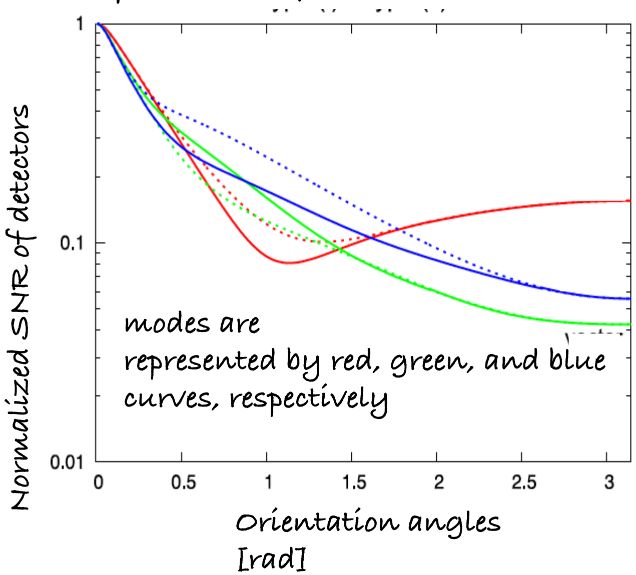
$$\mathcal{N} = N(N-1)/2$$

gravitational degrees of freedom are allowed.

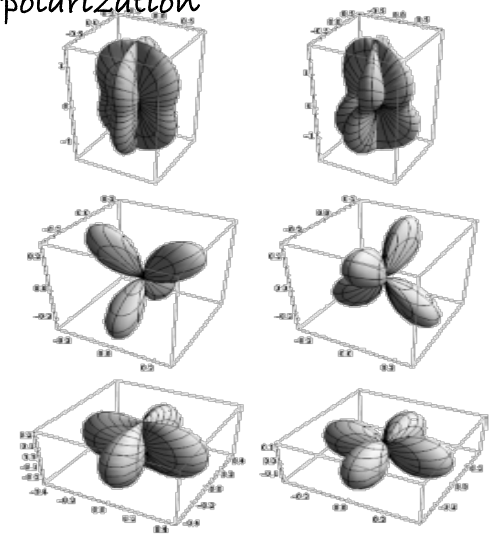
Detector response to stochastic background of GWs



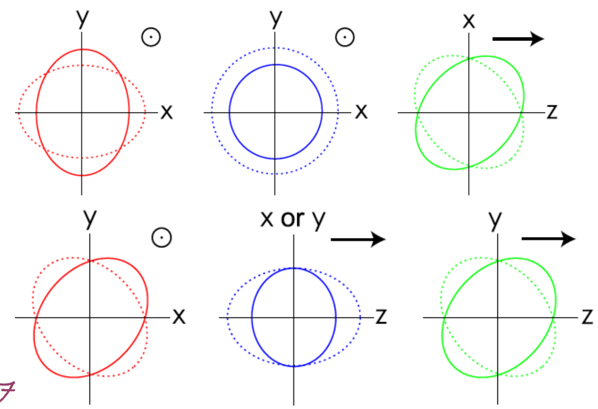
We have investigated the possible detectability of such additional polarization modes of a stochastic gravitational wave by ground-based and space interferometric detectors.



Plots of angular pattern functions of a detector for each polarization



Displacement induced by each mode on a sphere of test particles



We found that these massive modes are certainly of interest
for direct detection by the VIRGO-LIGO, LISA experiments.

S. Capozziello, R. Cianci, M. De Laurentis, S. Vignolo, EPJC 70, 341-349, (2010)

S. Bellucci, S. Capozziello, M. De Laurentis, V. Faraoni, Phys. Rev. D 79, 104004 (2009)

S.Capozziello, C. Corda, M. De Laurentis, MPLA 22, 2647, (2007); MPLA 15, 1097, (2007

Quadrupolar gravitational radiation in f(R)-gravity

We calculate the Minkowskian limit for a class of analytic f(R)-Lagrangian

$$f(R) = \sum_n \frac{f^n(R_0)}{n!}(R-R_0)^n \simeq f_0 + f_0' R + \frac{1}{2} f_0'' R^2 + \cdots$$

Field equations at the first order of approximation in term of the perturbation, become:

$$f_0'\left[R^{(1)}_{\mu\nu} - \frac{R^{(1)}}{2}\eta_{\mu\nu}\right] - f_0''\left[R^{(1)}_{,\mu\nu} - \eta_{\mu\nu}\Box R^{(1)}\right] = \frac{\mathcal{X}}{2} T^{(0)}_{\mu\nu}$$

The explicit expressions of the Ricci tensor and scalar, at the first order in the metric perturbation, read

$$\begin{cases} R^{(1)}_{\mu\nu} = h^{\sigma}_{(\mu,\nu)\sigma} - \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2} h_{,\mu\nu} \\ R^{(1)} = h^{,\sigma\tau}_{\sigma\tau} - \Box h \end{cases}$$

M. De Laurentis, I. De Martino IJGMMP 12, 1550004 (2014)

M. De Laurentis, I. De Martino MNRAS 431, 741 (2013)

M. De Laurentis, S. Capozziello, Astrop. Phys. 35, 5, 257 (2011)

Quadrupolar gravitational radiation in f(R)-gravity

Assuming that the source is localized in a finite region, as a consequence, outside this region

$$\longrightarrow \quad T_{\mu\nu} = 0 \quad \longrightarrow \quad R^{(1)}_{\mu\nu} = \Box h_{\mu\nu} = 0$$

the energy momentum tensor of gravitational field in f (R) gravity

$$t^\lambda_\alpha = f'\left\{\left[\frac{\partial R}{\partial g_{\rho\sigma,\lambda}} - \frac{1}{\sqrt{-g}}\partial_\xi\left(\sqrt{-g}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}\right)\right]g_{\rho\sigma,\alpha} + \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}\right\} - f'' R_{,\xi}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\alpha} - \delta^\lambda_\alpha\ f$$

the energy momentum tensor consists of a sum of a GR contribution plus a term coming from f (R) gravity:

$$t^\lambda_\alpha = f'_0 t^\lambda_{\alpha\,|\mathrm{GR}} + f''_0 t^\lambda_{\alpha\,|f(R)}$$

which in terms of the perturbation h is

$$t^\lambda_\alpha \sim f'_0 t^\lambda_{\alpha|\mathrm{GR}} + f''_0\Big\{(h^{\rho\sigma}_{,\rho\sigma} - \Box h)\Big[h^{\lambda\xi}_{,\xi\alpha} - h^{,\lambda}_{\alpha} - + \frac{1}{2}\delta^\lambda_\alpha(h^{\rho\sigma}_{,\rho\sigma} - \Box h)\Big]$$
$$-h^{\rho\sigma}_{,\rho\sigma\xi}h^{\lambda\xi}_{,\alpha} + h^{\rho\sigma}_{,\rho\sigma}{}^{,\lambda}h_{,\alpha} + h^{\lambda\xi}_{,\alpha}\Box h_{,\xi} - \Box h^{,\lambda}h_{,\alpha}\Big\}.$$

the energy momentum tensor assumes the following form:

$$\longrightarrow \quad t^\lambda_\alpha = \underbrace{f'_0 k^\lambda k_\alpha\left(\dot{h}^{\rho\sigma}\dot{h}_{\rho\sigma}\right)}_{GR} - \underbrace{\frac{1}{2}f''_0\delta^\lambda_\alpha\left(k_\rho k_\sigma \ddot{h}^{\rho\sigma}\right)^2}_{f(R)}$$

Radiated Energy

In order to calculate the radiated energy of a GW source suppose that $h_{\mu\nu}$ can be represented by a discrete spectral representation.

The instantaneous flux of energy is given by $$\frac{dE}{dt} = r^2 d\Omega \hat{x}^i t^{0i}$$

Defining the following momenta of the mass-energy distribution:

$$M(t) \simeq \int d^3\vec{x}\, T^{00}(\vec{x},t),$$

$$D^k(t) \simeq \int d^3\vec{x}\, x^k T^{00}(\vec{x},t),$$

$$Q^{ij}(t) \simeq \int d^3\vec{x}\, x^i x^j T^{00}(\vec{x},t).$$

and analysing the radiation in terms of multipoles, found

$$\langle t^\lambda_\alpha \rangle = \Bigg\langle f'_0 k^\lambda k_\alpha \frac{4}{r^2}\left[\left(\hat{x}_i\hat{x}_j\dddot{Q}^{ij}\right)^2 - 2\left(\hat{x}_k\dddot{Q}^{ik}\right)\left(\hat{x}_j\dddot{Q}^{ij}\right) + \left(\dddot{Q}^{ij}\dddot{Q}_{ij}\right)\right]$$
$$-f''_0\delta^\lambda_\alpha (k_\rho k_\sigma)^2 \frac{2}{r^2}\left[\left(\hat{x}_i\hat{x}_j\dddot{Q}^{ij}\right)^2 + -2\left(\hat{x}_k\dddot{Q}^{ik}\right)\left(\hat{x}_j\dddot{Q}^{ij}\right) + \left(\dddot{Q}^{ij}\dddot{Q}_{ij}\right)\right]\Bigg\rangle$$

the total average flux of energy due to the tensor wave

$$\underbrace{\left\langle \frac{dE}{dt}\right\rangle}_{(total)} = \frac{G}{60}\left\langle \underbrace{f'_0\left(\dddot{Q}^{ij}\dddot{Q}_{ij}\right)}_{GR} - \underbrace{f''_0\left(\dddot{Q}^{ij}\dddot{Q}_{ij}\right)}_{f(R)}\right\rangle$$

Precisely, for $f''_0 = 0$ and $f'_0 = 4/3$ $\Longrightarrow$

$$\underbrace{\left\langle \frac{dE}{dt}\right\rangle}_{(GR)} = \frac{G}{45}\left\langle \dddot{Q}^{ij}\dddot{Q}_{ij}\right\rangle$$

Application to the binary systems

Our goal is to use a sample of binary pulsar systems to fix bounds on $f(R)$ parameters.

We assume that the motion is Keplerian and the orbit is in the (x, y) plane

the quadrupole matrix is $Q_{ij} = \mu r^2 \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{pmatrix}_{ij}$

whit $\dot{\psi} = \left(\frac{G m_c}{a^3}\right)^{1/2} \left(1-\epsilon^2\right)^{-3/2} (1+\epsilon\cos\psi)^2$

the time derivatives of the quadrupole:

$$\dddot{Q}_{11} = \mathcal{H}_1 \sin 2\psi (\epsilon\cos\psi + 1)^2 (3\epsilon\cos\psi + 4)$$

$$\dddot{Q}_{22} = -\mathcal{H}_1 (8\cos\psi + \epsilon(3\cos 2\psi + 5)) \times \sin\psi(\epsilon\cos\psi + 1)^2,$$

$$\dddot{Q}_{12} = -\mathcal{H}_1(\epsilon\cos\psi + 1)^2 \times (5\epsilon\cos\psi + 3\epsilon\cos 3\psi + 8\cos 2\psi)$$

$$\ddddot{Q}_{11} = \mathcal{H}_2 \left[15\epsilon^2\cos 4\psi + 50\epsilon\cos 3\psi + \left(12\epsilon^2 + 32\right)\cos 2\psi + 6\epsilon\cos\psi - 3\epsilon^2\right]$$

$$\ddddot{Q}_{22} = -\mathcal{H}_2 \left[15\epsilon^2\cos 4\psi + 50\epsilon\cos 3\psi + \left(24\epsilon^2 + 32\right)\cos 2\psi + 14\epsilon\cos\psi - 7\epsilon^2\right]$$

$$\ddddot{Q}_{12} = 2\mathcal{H}_2 \sin\psi \left[15\epsilon^2\cos 3\psi + 50\epsilon\cos 2\psi + \left(33\epsilon^2 + 32\right)\cos\psi + 30\epsilon\right],$$

where

$$\mathcal{H}_1 = \frac{(2\pi)^{5/3} G^{2/3} m_c m_p}{T^{5/3}\left(1-\epsilon^2\right)^{5/2} \sqrt[3]{m_c + m_p}},$$

$$\mathcal{H}_2 = \frac{2^{2/3}\pi^{8/3} G^{2/3} m_c m_p (\epsilon\cos\psi + 1)^3}{T^{8/3}\left(\epsilon^2 - 1\right)^4 \sqrt[3]{m_c + m_p}}$$

Application to the binary systems

we can perform the time average of the radiated power by writing

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle = \frac{1}{T}\int_0^T \mathrm{d}t\, \frac{\mathrm{d}E(\psi)}{\mathrm{d}t} = \frac{1}{T}\int_0^{2\pi} \frac{\mathrm{d}\psi}{\dot{\psi}}\frac{\mathrm{d}E(\psi)}{\mathrm{d}t}$$

and finally, we get the first time derivative of the orbital period:

$$\begin{aligned}\dot{T}_{\mathrm{b}} = &-\frac{3}{20}\left(\frac{T}{2\pi}\right)^{-5/3}\frac{\mu G^{5/3}(m_{\mathrm{c}}+m_{\mathrm{p}})^{2/3}}{c^5(1-\epsilon^2)^{7/2}} \\ &\times \left[f_0' \left(37\epsilon^4 + 292\epsilon^2 + 96\right) - \frac{f_0''\pi^2 T^{-1}}{2(1+\epsilon^2)^3} \right. \\ &\left. \times \left(891\epsilon^8 + 28\,016\epsilon^6 + 82\,736\epsilon^4 + 43\,520\epsilon^2 + 3072\right)\right]\end{aligned}$$

we will go on to constrain the $f(R)$ theories estimating f''_0 from the comparison between the theoretical predictions of dT_b and the observed one.

M. De Laurentis, I. De Martino IJGMMP 12, 1550004 (2014)

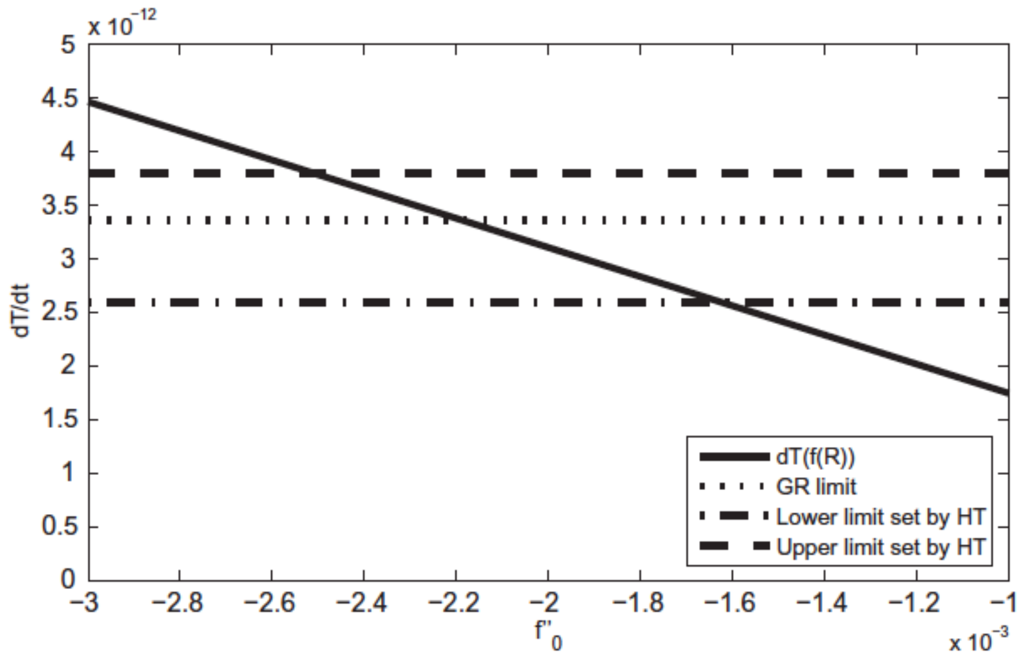
M. De Laurentis, I. De Martino MNRAS 431, 741 (2013)

M. De Laurentis, R. De Rosa, F. Garufi and L. Milano, Mon. Not. R. Astron. Soc. 424, 2371 (2012)

Application to the binary systems: The PSR 1913 + 16 case

Let us now use the published numerical values for the specific example of PSR 1913 + 16 to numerically evaluate the above equations

PSR 1913 + 16	Chacteristic features
Pulsar mass	$m = 1.39 M_{\odot}$
Companion mass	$M = 1.44 M_{\odot}$
Inclination angle	$\sin i = 0.81$
Orbit semimajor axis	$a = 8.67 \times 10^{10}$ cm
Eccentricity	$\epsilon = 0.617155$
Gravitational constant	$G = 6.67 \times 10^{-8}$ dyn cm^2 g^{-2}
Speed of light	$c = 2.99 \times 10^{10}$ cm s^{-1}



Orbital decay rate for PSR 1913 + 16 in $f(R)$-gravity. Upper limit set by Taylor et al. in dashed line. GR limit 3.36×10^{-12} in dotted line and the lower limit set by Taylor et al. in dashdot line. While in solid line is plotted $dT_{f(R)}$

A class of $f(R)$ agrees with data!

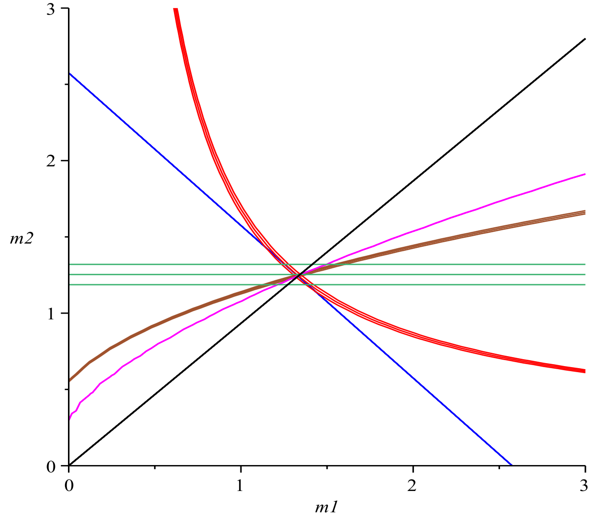
R.A. Hulse, J.M. Taylor ApJ Lett. 195 L51 (1975)
J.H. Taylor, L.A. Flower, P.M. Mc Culloch Nature 277 437 (1979) ;
J.H. Taylor, J.M. Weisberg, Astrophys J. 253 , 908 (1982)

Application to the binary systems: PPK parameters for PSR J0737-3039

$$\dot{\omega} = \left(\frac{2\pi}{P_b}\right)^{5/3} \frac{G_{AB}^{2/3}(m_1+m_2)^{2/3}}{c^2(1-e^2)} \left[(2\gamma^{PPN}+1) - \right.$$
$$\left. -\frac{1}{2}\,\frac{m_1(2\beta^{PPN}-1)\,G^2 + m_2(2\beta^{PPN}-1)\,G^2}{G_{AB}^2(m_1+m_2)} + \frac{1}{2}\right],$$
$$\gamma = e\left(\frac{2\pi}{P_b}\right)^{-1/3} \frac{m_2}{m_1+m_2} \times \left(G_{02} + \frac{G_{AB} m_2}{m_1+m_2} + k\eta *\right) \times$$
$$\times \frac{G_{AB}^{-1/3}\,(m_1+m_2)^{2/3}}{c^2},$$
$$r = \frac{G_{02}}{4c^3}(1+\varepsilon_{02})\,m_2,$$
$$s = \left(\frac{2\pi}{P_b}\right)^{2/3} \frac{cx(m_1+m_2)^{2/3}}{G_{AB}^{1/3}\,m_2},$$

$$\gamma_R^{PPN} - 1 = -\frac{f''(R)^2}{f'(R)+2f''(R)^2},$$

$$\beta_R^{PPN} - 1 = \frac{1}{4}\,\frac{f'(R)f''(R)}{2f'(R)^2+3f''(R)^2}\,\frac{d\gamma_R^{PPN}}{dR}$$



Dependence of the companion mass upon the pulsar Colors indicate:
Curve $\omega(m_1,m_2)$ is blue, curve $\gamma(m_1,m_2)$ is brown, curve $P_b(m_1,m_2)$ is red, curve s (m_1,m_2) is pink, curve $r(m_1,m_2)$ is green, curve $R(m_1,m_2)$ is black.

In GR we have the following masses for PSR J0737-3039

$$m_{p1} = 1.3381,$$
$$m_{p2} = 1.2489.$$

In f(R) we obtain

$$m_{p1} = 1.3331,$$
$$m_{p2} = 1.2429.$$

M. De Laurentis, I. De Martino, P. Freire in preparation

Modified TOV equations in f(R) gravity

the equations for a spherically symmetric and static perfect fluid also in f(R) gravity

$$\frac{f'(R)}{r^2}\frac{d}{dr}\left[r\left(1-e^{-2\lambda}\right)\right] =$$
$$\frac{8\pi\rho}{c^4}+\frac{1}{2}\left[f'(R)R-f(R)\right]$$
$$+e^{-2\lambda}\left[\left(\frac{2}{r}-\frac{d\lambda}{dr}\right)\frac{df'(R)}{dr}+\frac{d^2f'(R)}{dr^2}\right]$$

and

$$\frac{f'(R)}{r}\left[2e^{-2\lambda}\frac{d\phi}{dr}-\frac{1}{r}\left(1-e^{-2\lambda}\right)\right] =$$
$$\frac{8\pi p}{c^4}+\frac{1}{2}\left[f'(R)R-f(R)\right]+e^{-2\lambda}\left(\frac{2}{r}+\frac{d\phi}{dr}\right)\frac{df'(R)}{dr}$$

we need a further equations to solve the above system and then we consider also the trace equation in the following form:

$$3\Box f'(R)+f'(R)R-2f(R)=-8\pi(\rho-3p)$$

remembering that $$e^{2\lambda}\Box=-e^{(2\lambda-2\phi)}\frac{\partial^2}{\partial t^2}+\left(\frac{2}{r}+\frac{d\phi}{dr}-\frac{d\lambda}{dr}\right)\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}$$

which for $f(R) = R$ is reduced to the equality $R = 8\pi(\rho - 3p)$

The case of $f(R) = R + R\epsilon \log R$

Let us consider a correction to the Hilbert-Einstein action given by

$$\boxed{f(R) = R^{1+\epsilon}}$$

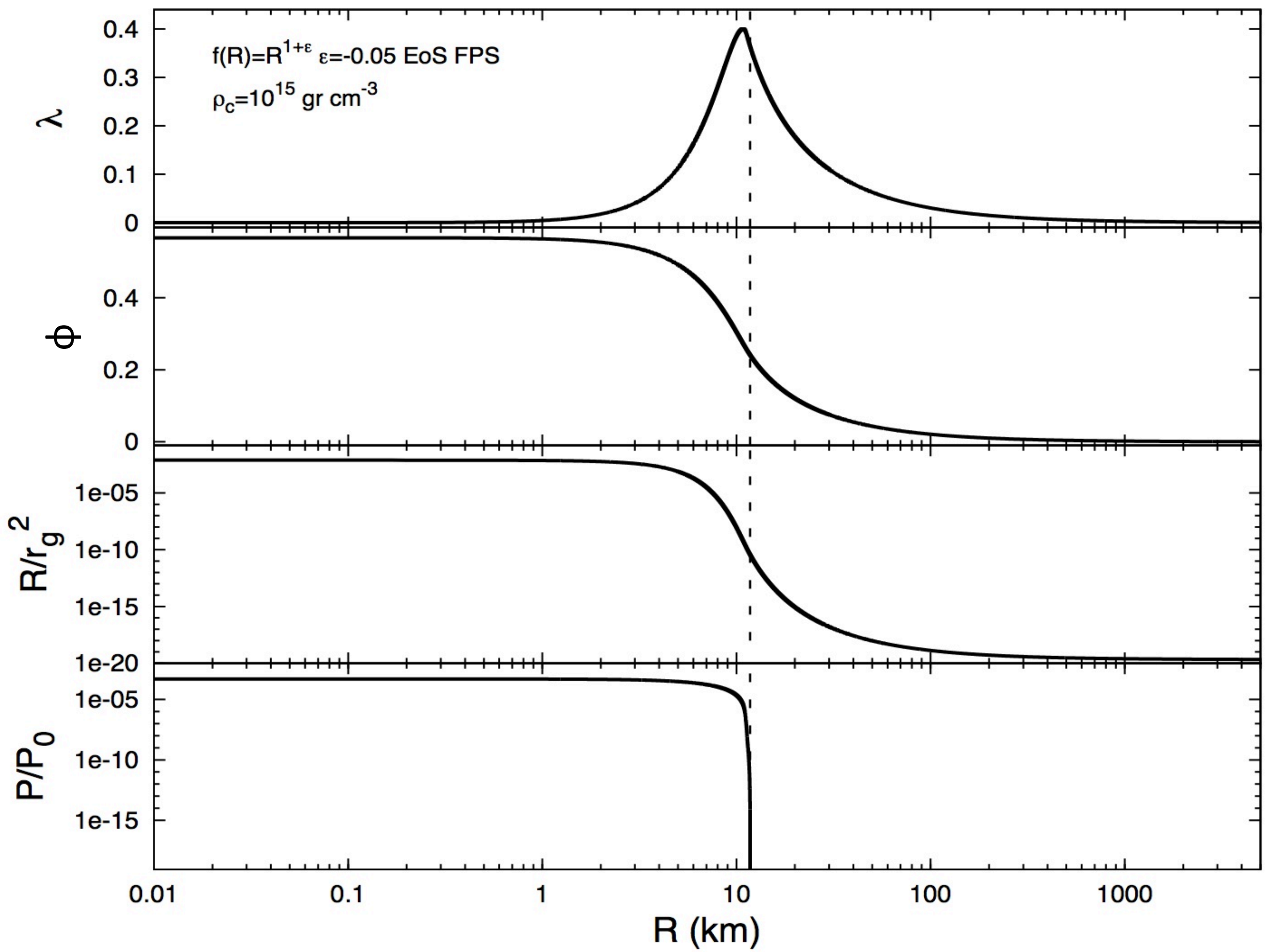
It is easy to show that

$$\begin{aligned} R^{1+\epsilon} &= R \cdot R^{\epsilon} \simeq R(1 + (\log R)\epsilon + \mathcal{O}(\epsilon^2)) \\ &\simeq R + \epsilon R \log R\,, \end{aligned}$$

It is interesting to define the right physical dimensions of the coupling constant and to control the magnitude of the corrections with respect to the standard Einstein gravity

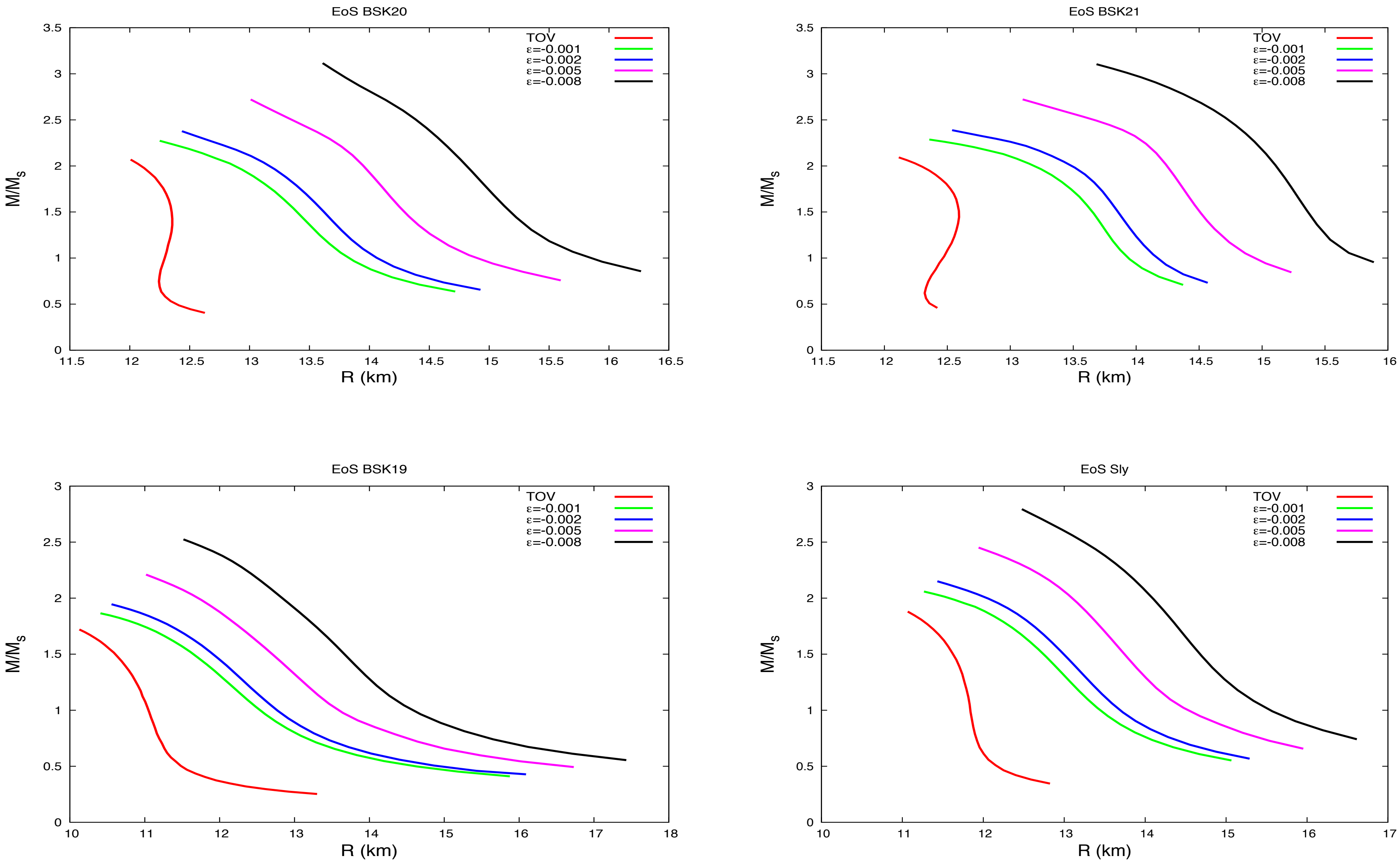
S. Capozziello, M. De Laurentis, M. Francaviglia, Astrop. Phys. 29, 125 (2008)
M. De Laurentis, R. De Rosa, F. Garufi and L. Milano, Mon. Not. R. Astron. Soc. 424, 2371 (2012)
T. Clifton, J. D. Barrow, Phys. Rev. D 72 103005 (2005)
T. Clifton, J. D. Barrow, Phys. Rev. D 81, 063006 (2010)
S. Capozziello, A. Stabile, A. Troisi, Class. Quantum Grav. 25 085004 (2008)
S. Capozziello, M. De Laurentis, A. Stabile Class. Quantum Grav. 27, 165008 (2010)

Example of solution of the field equations



S. Capozziello, M. De Laurentis, R. Farinelli, S.D. Odintsov arXiv: 1509.04163

M-R diagram



For each EOS the maximal central density is determined by the condition $\rho_c - 3p > 0$

S. Capozziello, M. De Laurentis, R. Farinelli, S.D. Odintsov arXiv: 1509.04163

Conclusions and perspectives

- ✓ ETGs are a useful approach to IR and UV problems of GR
- ✓ Naturally address problems like DE and DM extending the gravitational sector.
- ✓ However results of GR are easily recovered since Hilbert-Einstein action is just a particular ETG
- ✓ An important challenge is to find out exact solutions for ETGs. This allows to control mathematics and physics of the theory

- ✓ The general philosophy is that gravity could not be the same at any scale and GR is a good theory only at scales investigated up to now

- ✓ We are searching for an EXPERIMENTUM CRUCIS to retain definitely such theories or rule out them

Black Hole Cam project can give hints in this direction.....

Work in progress!!!

Hints are welcome!!!