Observations on past null cones, numerical relativity, and cosmology

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Outline and key references

- Bondi-Sachs formalism in numerical relativity

  NT Bishop, R Gomez, L Lehner, M Maharaj and J Winicour *High-powered gravitational news* Phys. Rev. D **56** 6298-6309 (1997)

- Characteristic extraction

  N.T. Bishop , R Gomez, L Lehner, B Szilagyi and J Winicour *Cauchy-characteristic matching* in B. Iyer and B. Bhawal (Eds.) *Black Holes, Gravitational Radiation and the Universe* Kluwer, Dordrecht, The Netherlands (1999)

  C Reisswig, NT Bishop, D Pollney and B Szilagyi, *Characteristic extraction in numerical relativity: binary black hole merger waveforms at* $\mathcal{J}^+$, Class. Quantum Grav. **27** 075014 (2010)

- Cosmology

  PJ van der Walt and NT Bishop, *Observational cosmology using characteristic numerical relativity*, Phys. Rev. D **82** 084001 (2010)

  PJ van der Walt and NT Bishop, *Observational cosmology using characteristic numerical relativity: Characteristic formalism on null geodesics*, Phys. Rev. D *85* 044016 (2012)

  HL Bester, J Larena, PJ van der Walt and NT Bishop, *What's inside the cone? Numerically reconstructing the metric from observations*, JCAP **2** 009 (2014)

  HL Bester, J Larena and NT Bishop, *Towards the geometry of the Universe from data*, MNRAS **453** 23642377 (2015)

- Conclusion

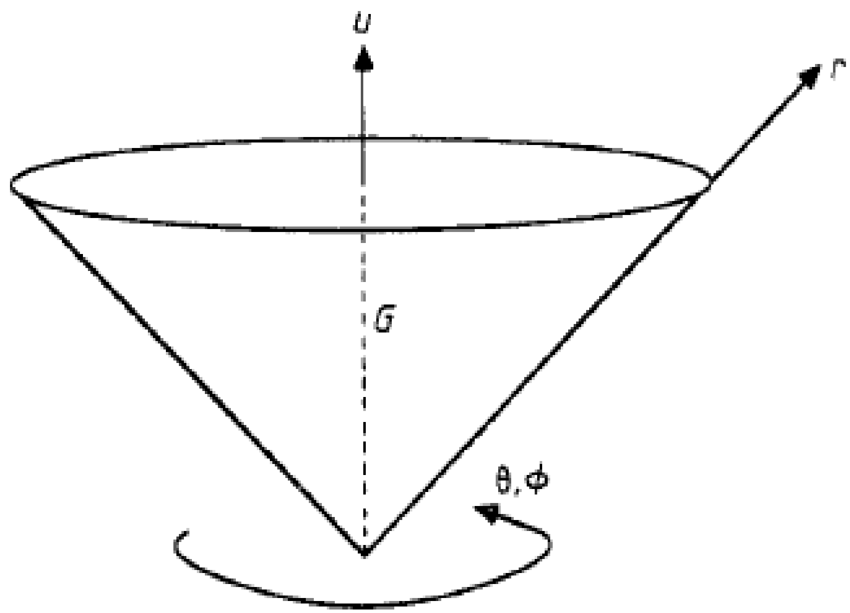


Figure 1. Null coordinates.

Background: Bondi-Sachs Formalism*

The Bondi-Sachs metric is

$$ds^2 = -\left(e^{2\beta}(1 + W_c r) - r^2 h_{AB} U^A U^B\right) du^2$$
$$-2e^{2\beta} du dr - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B,$$

where r is an area coordinate so that $det(h_{AB}) = det(q_{AB})$ with q_{AB} a unit sphere metric (e.g. $d\theta^2 + \sin^2\theta d\phi^2$). We introduce a complex dyad q_A (e.g. $q_A = (1, i\sin\theta)$). Then h_{AB}, U^A can be represented by

$$J = h_{AB} q^A q^B / 2\,, \quad U = U^A q_A\,.$$

*H Bondi et al., Proc. R. Soc. London **A269** 2152 (1962); RK Sachs, Proc. Roy. Soc. London **A270** 103126 (1962)

Einstein's equations $R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)$ can be categorized as

- Hypersurface equations, $R_{11}, q^A R_{1A}, h^{AB} R_{AB}$,

$$\begin{aligned}
\beta_{,r} &= f_1(J) \\
U_{,rr} &= f_2(J, \beta) \\
W_{c,r} &= f_3(J, \beta, U)
\end{aligned}$$

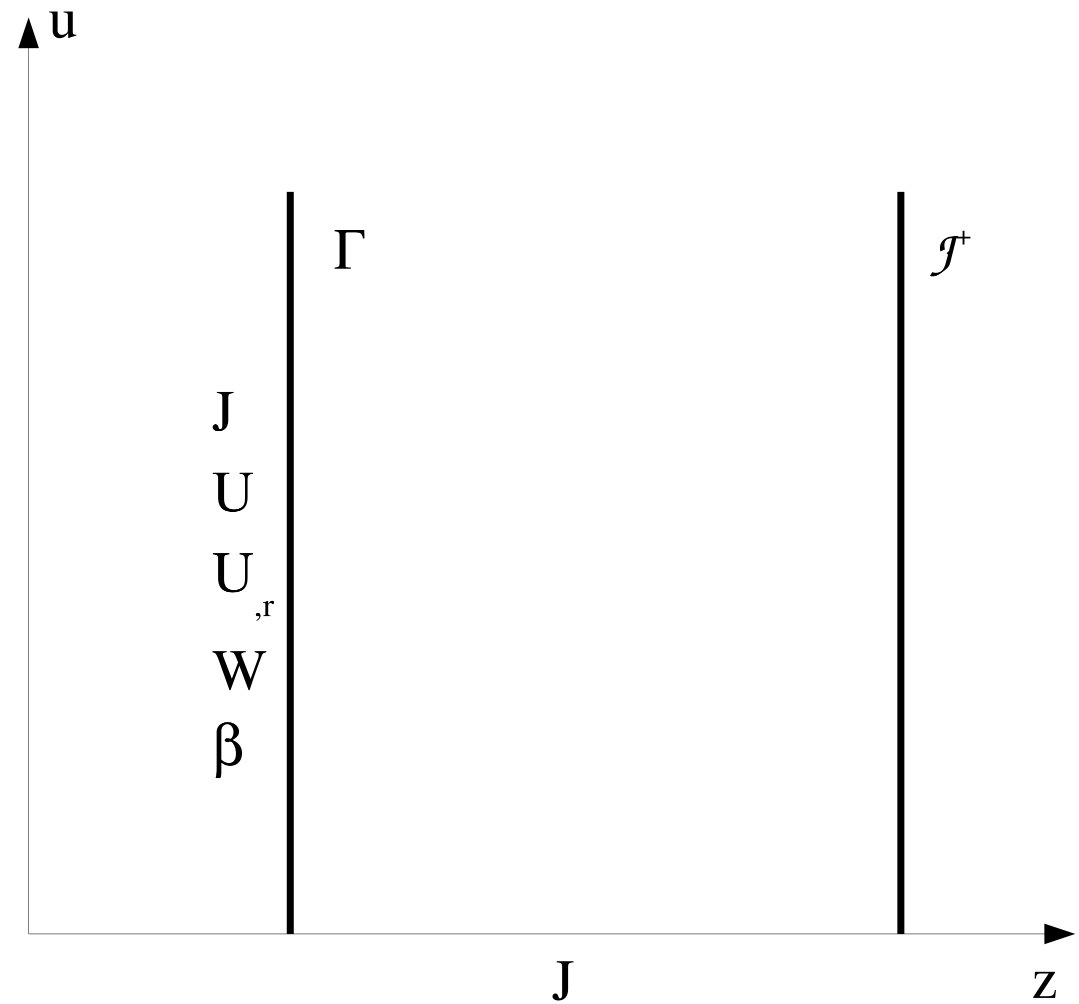
- Evolution equation $q^A q^B R_{AB}$

$$J_{,ur} = f_4(J, \beta, U, W_c)$$

  where the f_i include hypersurface derivatives (∂_r, ∂_A) of the variables

- Constraints $R_{0\alpha}$.

Compactify: $r \to x$ with $r = \infty \to x = 1$, e.g. $x = r/(1+r)$.


u
Γ
$\mathcal{J}^+$
J
U
$U_{,r}$
W
β
J
z

Inclusion of matter to the characteristic formalism[†]

The matter is characterized by

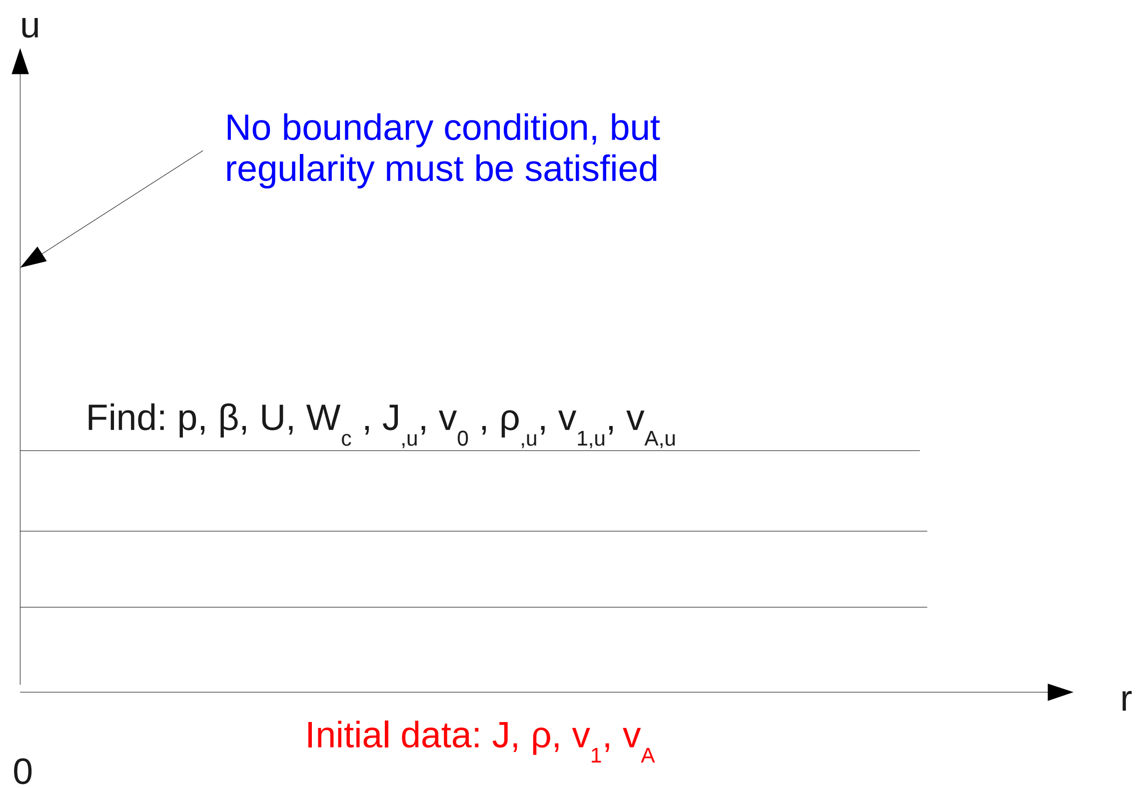
Density	ρ
Pressure	p
Covariant 4-velocity	$v_\alpha = (v_0, v_1, v_A)$
Equation of state	$p = p(\rho)$

Define $V = v_A q^A$, apply the Einstein equations, as well as the fluid conservation equations $T^a_{b;a} = 0$, to get

[†]N.T. Bishop *et al.*, Phys. Rev. D **60**, 024005 (1999)

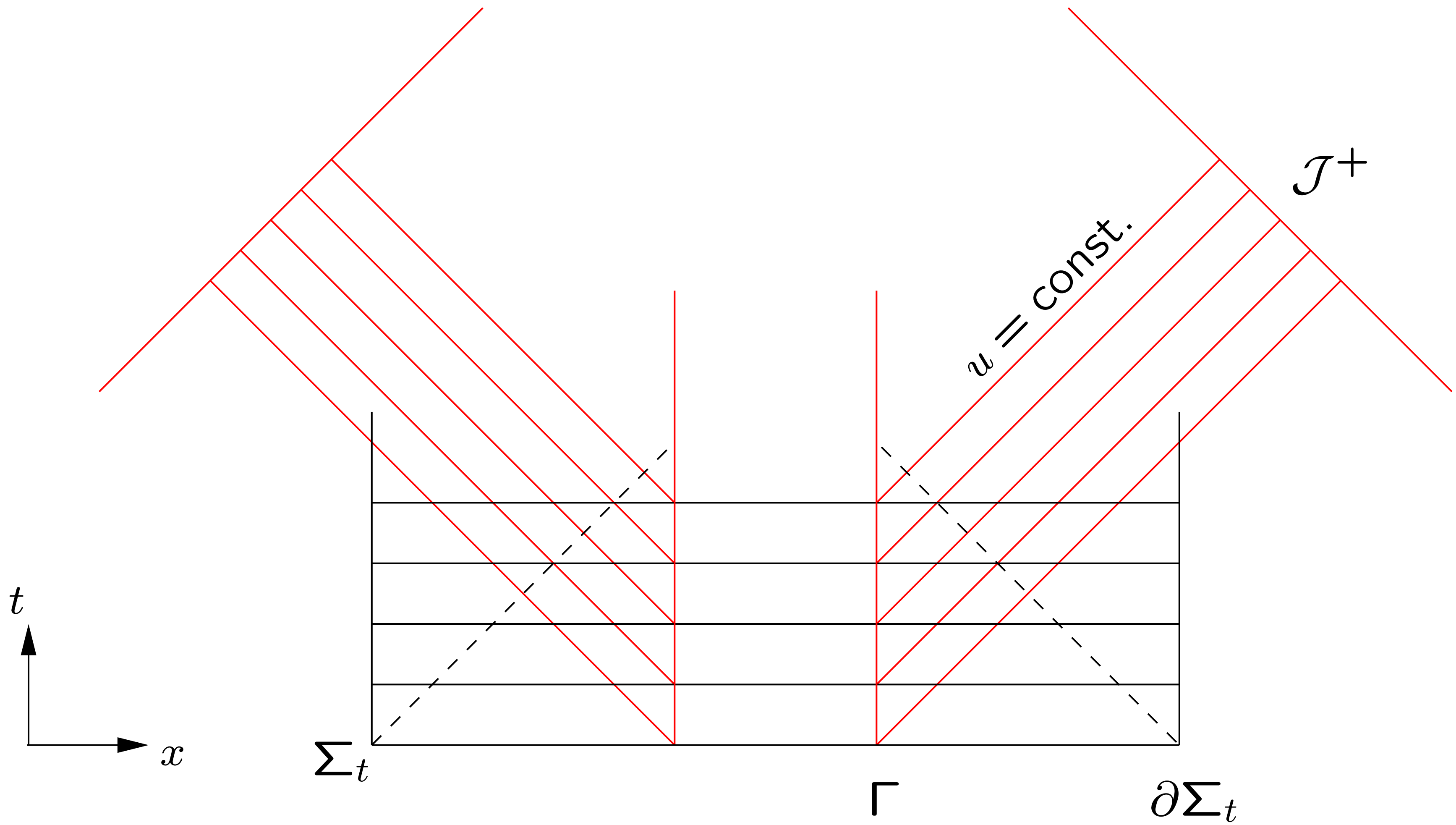
$$
\begin{aligned}
p &= p(\rho) \\
\beta_{,r} &= f_1 + 2\pi r(\rho + p)(v_1)^2 \\
U_{,rr} &= f_2 + (\rho + p)v_1 V F_2(r, \beta, J) \\
W_{c,r} &= f_3 + F_3(\rho, p, v_1, V, r, \beta, J) \\
J_{,ur} &= f_4 + F_4(\rho, p, V, r, \beta, J) \\
v_0 &= F_5(v_1, V, r, \beta, U, J) \\
\rho_{,u} &= F_6(\rho, p, , v_1, V, v_0, r, \beta, J, U, W_c) \\
v_{1,u} &= F_7(\rho, p, v_1, V, r, \beta, J, U, W_c) \\
V_{,u} &= F_8(\rho, p, v_1, V, v_0, r, \beta, J, U, W_c).
\end{aligned}
$$

Evolution variables	J, ρ, v_1, V
Auxiliary variables	p, β, U, W_c, v_0


u
No boundary condition, but
regularity must be satisfied
Find: p, β, U, W_c , $J_{,u}$, v_0 , $\rho_{,u}$, $v_{1,u}$, $v_{A,u}$
r
Initial data: J, ρ, v_1, v_A
0

Characteristic extraction

- Gravitational radiation is defined at future null infinity ($\mathcal{J}^+$)
- But ... It is extracted by perturbative matching, or from $r\psi_4$, using data at a finite distance from the source
- Characteristic numerical relativity has many positive features – stability, convergence, inclusion of null infinity
- Idea: use data on a finite worldtube as input to a characteristic code, and thereby calculate the radiation at $\mathcal{J}^+$



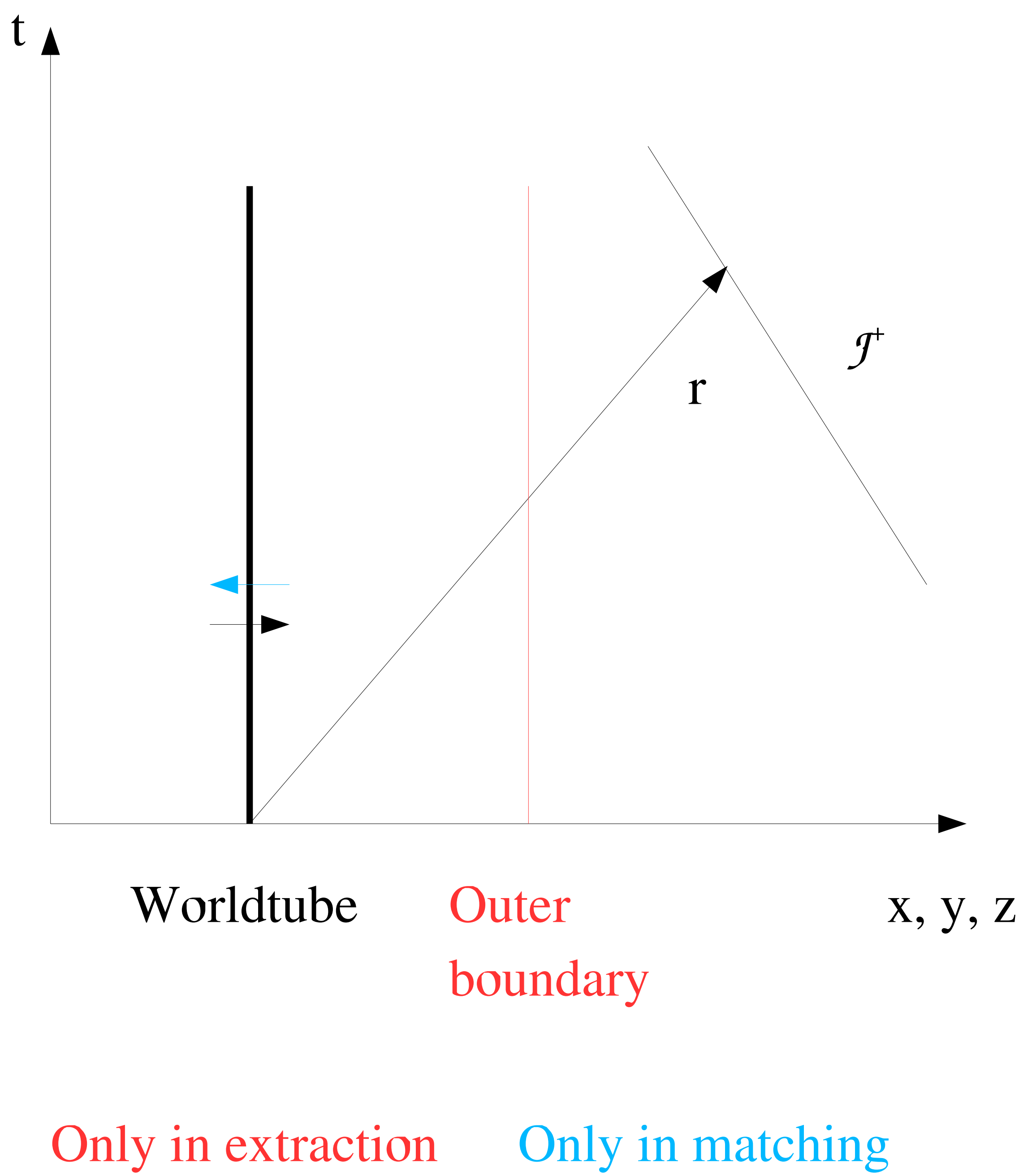
$\mathcal{J}^+$
$u = \text{const.}$
t
x
Σ_t
Γ
$\partial\Sigma_t$

Characteristic extraction and matching

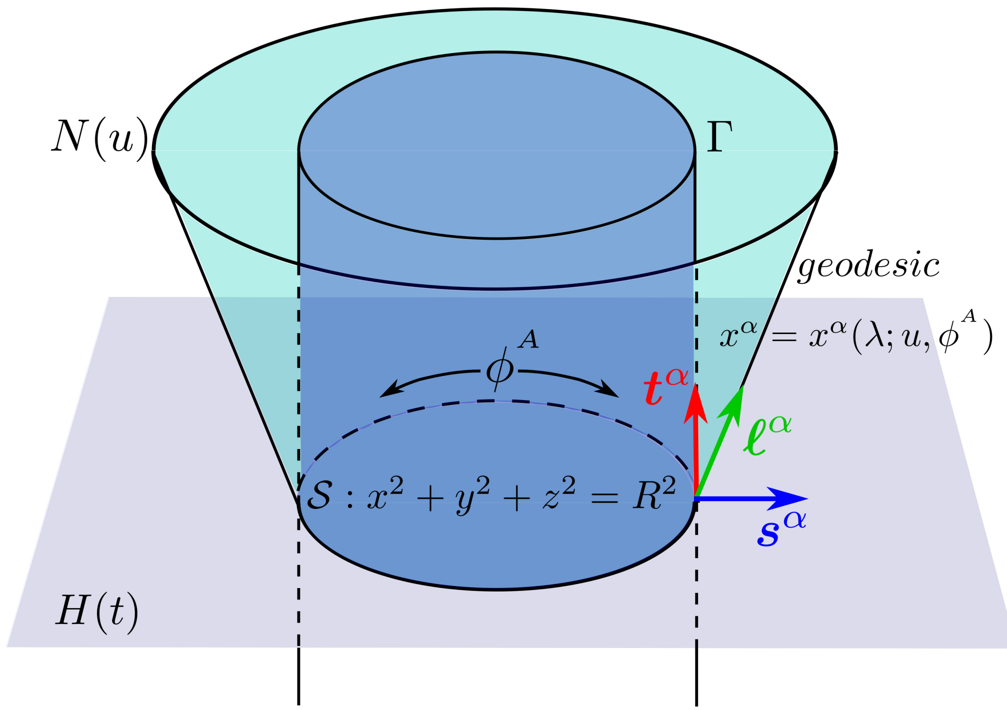
- Define a worldtube Γ, and use the Cauchy metric data to generate characteristic metric data on Γ; then use the characteristic Einstein equations to find metric data between Γ and $\mathscr{I}^+$, and compute the radiation at $\mathscr{I}^+$

- **In extraction:** Impose a standard outer boundary condition for the Cauchy evolution at some surface well outside Γ

- (In matching): Use characteristic metric data to provide an outer boundary condition for the Cauchy evolution


t
r
$\mathcal{J}^+$
Worldtube
Outer boundary
x, y, z
Only in extraction
Only in matching

Construction of characteristic boundary data



Step 1: Transform (t, x, y, z) to null affine coordinates (u, λ, ϕ^A).
Step 2: Define surface area coordinate r in terms of angular part of null affine metric, then transform $(u, \lambda, \phi^A) \to (u, r, \phi^A)$.

Gauge issues

The Bondi gauge naturally exhibits asymptotic flatness, and all Bondi-Sachs metric variables fall of as r^{-1} (or faster), so in the asymptotic limit the metric tends to the Minkowski form

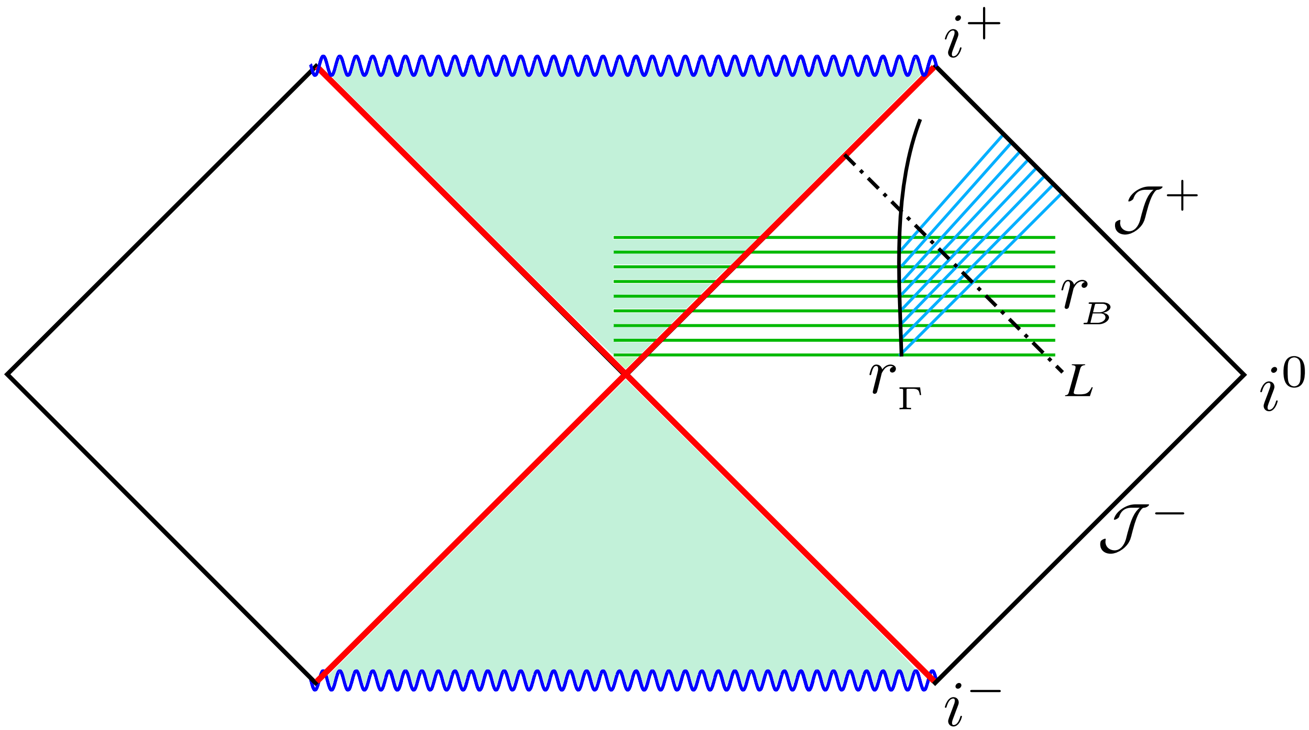
$$ds^2 = -du^2 - 2du\,dr + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,.$$

The various GW descriptors are ($\mathcal{N}$ is the Bondi news)

$$r\psi_4 = \partial_u^2(r\bar{J}),\ \mathcal{N} = \frac{\partial_u(rJ)}{2},\ r(h_+ + ih_\times) = (rJ),$$

evaluated in the limit as $r \to \infty$.

In practice, the gauge is not Bondi since it is fixed at the extraction worldtube not $\mathcal{J}^+$. A coordinate transformation, usually implicit, has to be made to the Bondi gauge. This is a complex process, and in the end the GW descriptors are given in terms of (general gauge) metric variables.



Error in GW extraction:

Unphysical initial Cauchy data

Unphysical outer boundary data if $r_B - r_\Gamma < t_{\text{final}} - t_{\text{initial}}$

Unphysical initial characteristic data

Truncation error, round-off error

COSMOLOGY

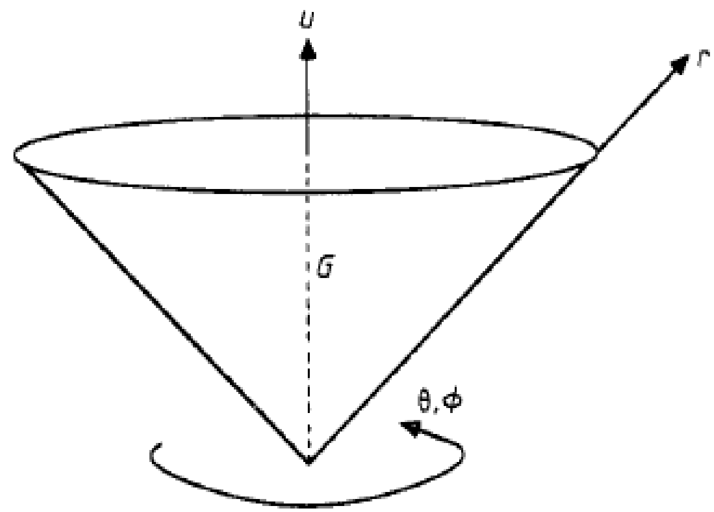


Figure 1. Null coordinates.

BECOMES

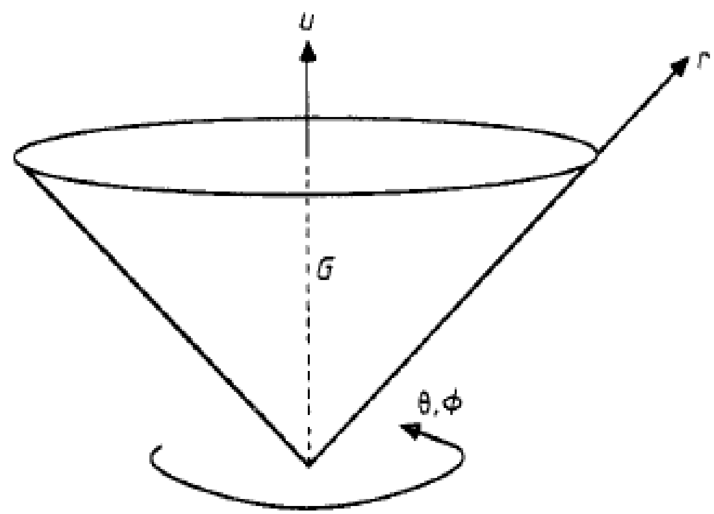


Figure 1. Null coordinates.

Observational cosmology

- Cosmological data is (almost all) from the past null cone, and it is natural to apply the characteristic approach.

- These ideas were developed by Kristian, Sachs, Ellis and others, using a different formalism. ‡

- Here, we adapt the characteristic code so that it actually computes the past behaviour of the Universe. We take the cosmological fluid as dust, with $p = 0$, $T_{ab} = \rho v_a v_b$.

‡I Kristian and RK Sachs, Astrophys. J. **143** 379 (1966) ;G.F.R. Ellis *et al.*, Phys. Rep. **124**, 315 (1985)

Data on the past null cone

- Initial data required by the code: J, ρ, v_1, V, as functions of r, x^A.

- Observational data

x^A	Position on the sky
d_L	Luminosity distance: related to r by $d_L = (1+z)^2 r$
z	Red-shift : $v^0 = 1 + z, v_1 = -e^{2\beta}(1+z)$
$\frac{dx^A}{du}$	Angular velocity : $v^A = (1+z)\dfrac{dx^A}{du}, v_A = F_9(r, z, \beta, U, J, \dfrac{dx^A}{du})$
n	Observed number count : $N = \dfrac{ne^{-2\beta}}{1+z}$proper number count
J	Shear: from observed shape of spherical object

- A relationship for ρ needs to be assumed, say $\rho = \alpha N$.

- The Einstein equation for R_{11} becomes

$$\beta_{,r} = f_1 + 2\pi r \frac{\alpha n e^{-2\beta}}{1+z}\left(-e^{2\beta}(1+z)\right)^2$$

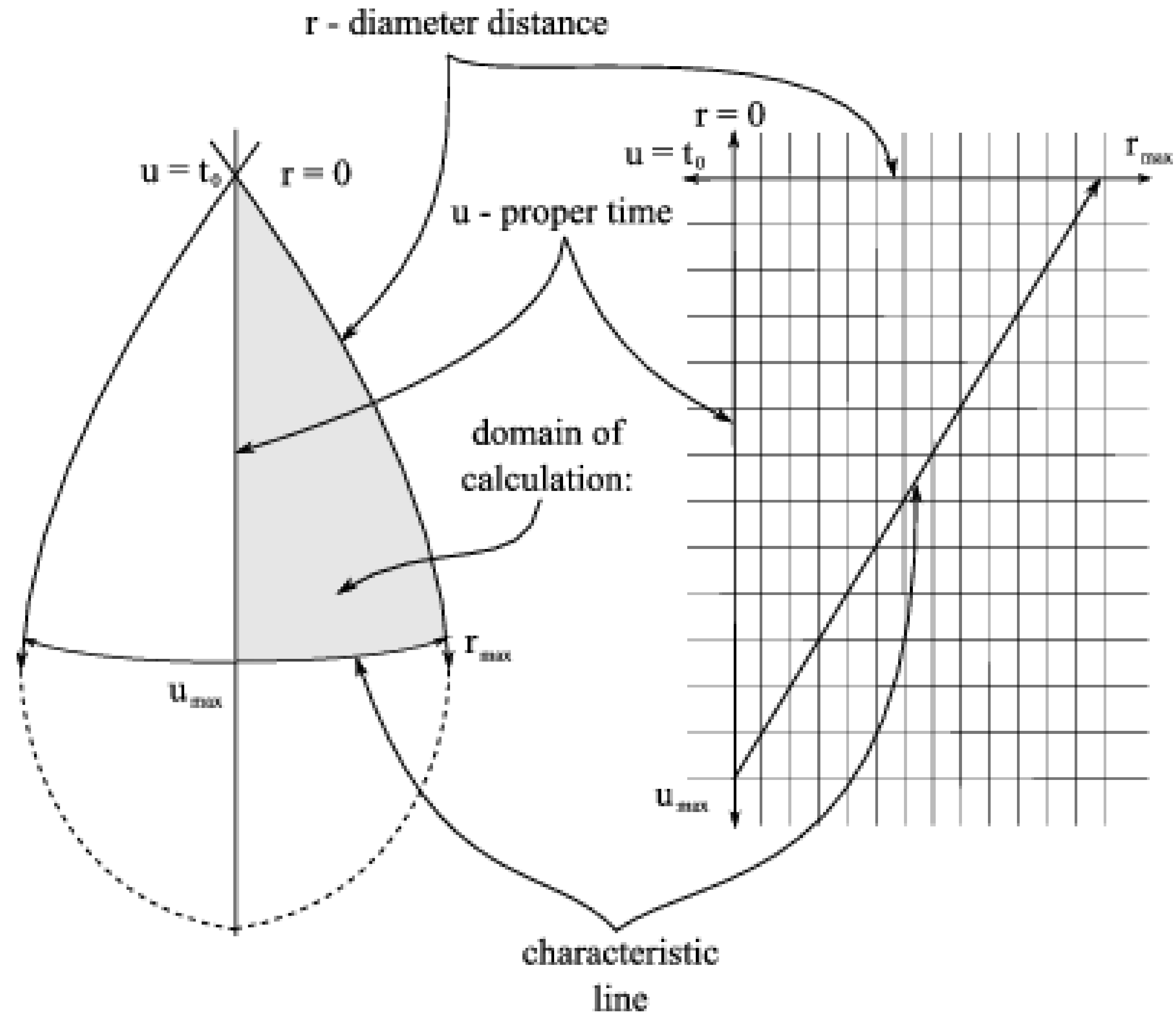
  which remains an o.d.e. for β. Once solved, v_1 is also found.

- Similarly, the R_{1A} equations remain o.d.es. for U, and once solved, v_A and hence V are also found.

Spherical symmetry

- Cosmology code implemented and tested for spherical symmetry.

- Difficult numerical issues
  - Evolution near the origin
  - Outer boundary – incoming null geodesic.



r - diameter distance
r = 0
u = t₀
r = 0
r_max
u - proper time
domain of calculation:
r_max
u_max
u_max
characteristic line

Code testing

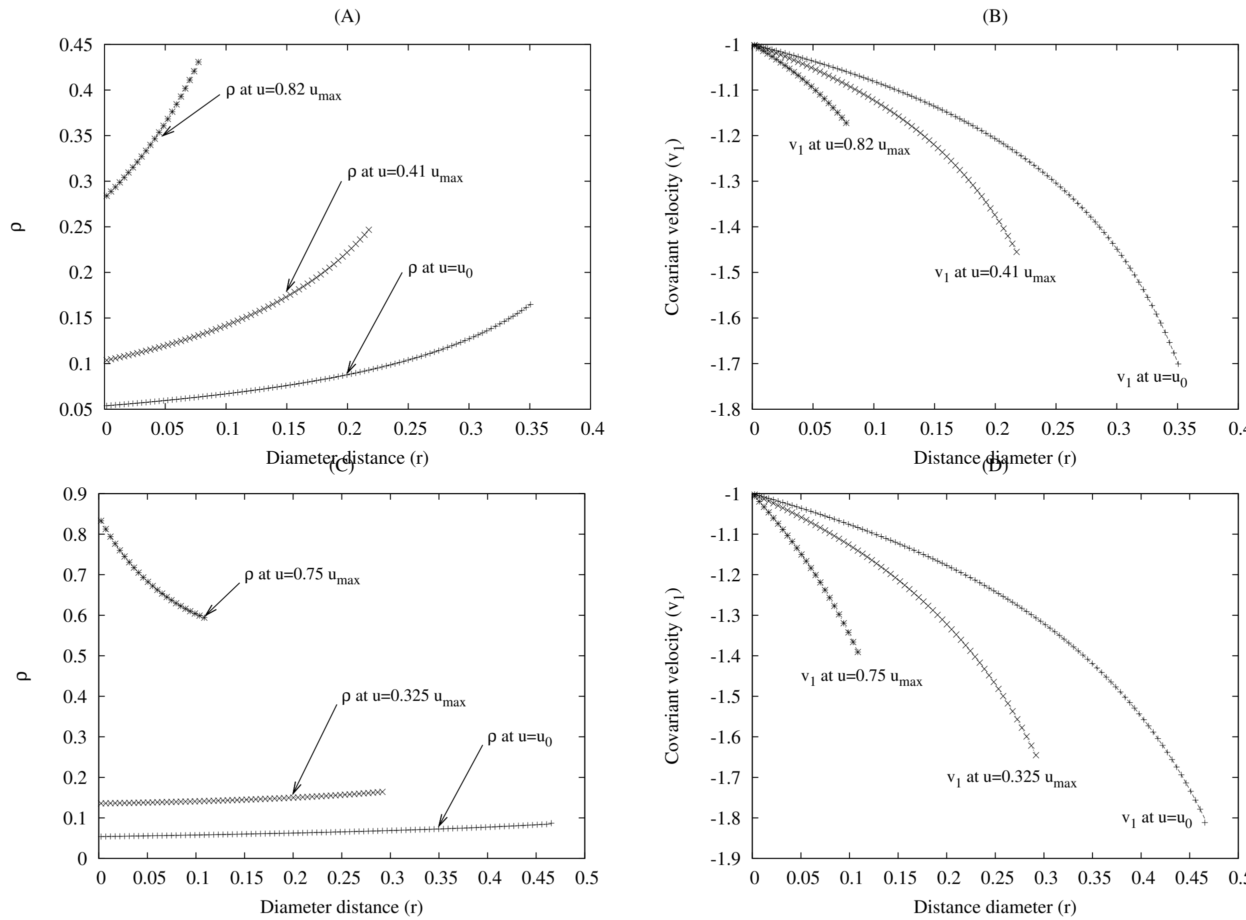
- Use Lemaître-Tolman-Bondi (LTB) model as exact solution

$$ds^2 = -dt^2 + [R_{,r}(t,r)]^2 dr^2 + [R(t,r)]^2 (d\theta^2 + sin^2\theta d\phi^2).$$

- Spherically symmetric inhomogeneous model, with FRW recovered when $R(t,r) = ra(t)$.

- We use $R(t,r) = r(t - br)^{2/3}$, so $b = 0$ is Einstein-de Sitter.

- Construct (numerically) coordinate transformation to characteristic coordinates.

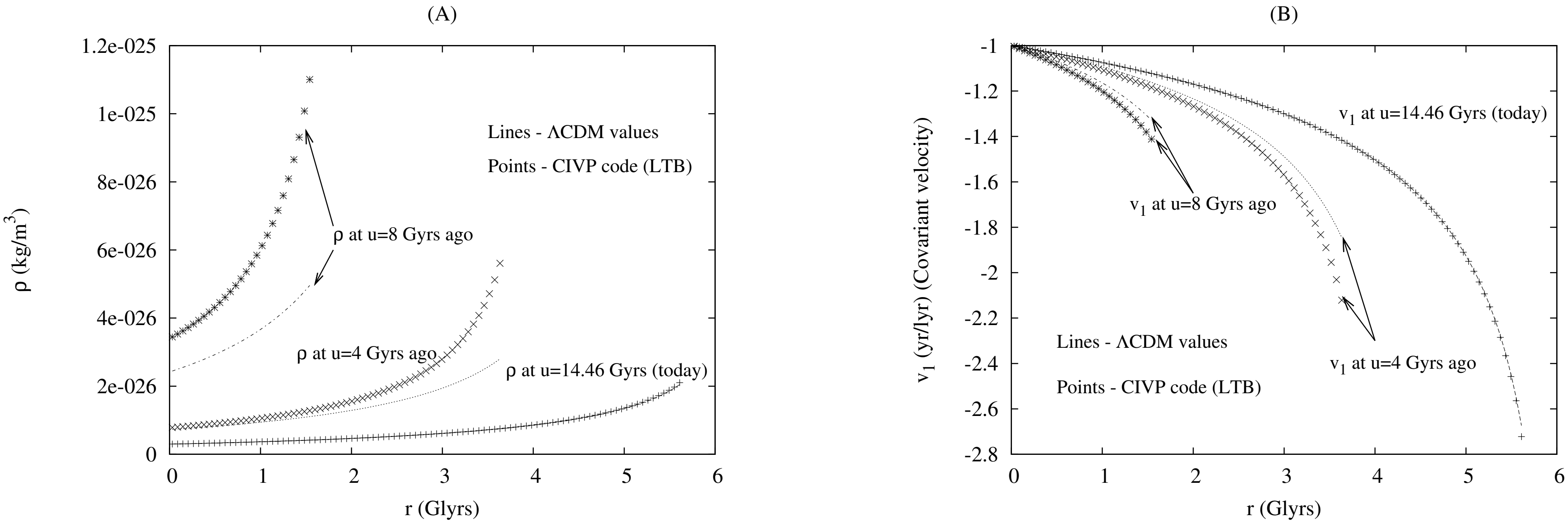


(A), (B): EdS; (C), (D): LTB with $b = -0.5$

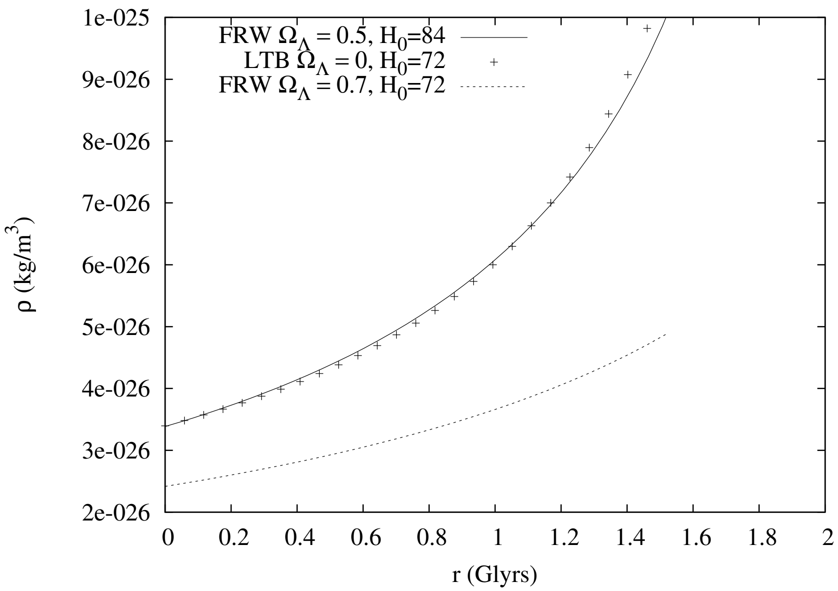
LTB vs ΛCDM

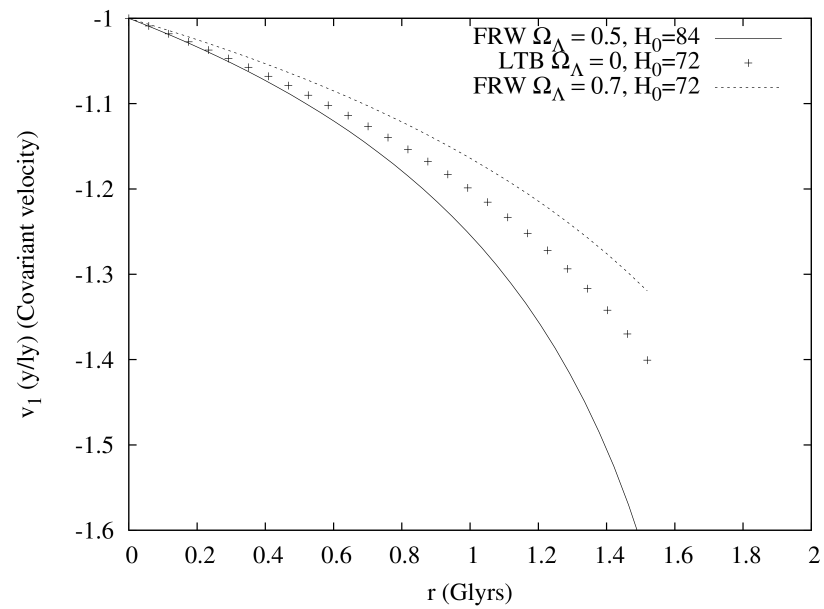
- SnIa data is usually regarded as caused by dark energy, but could also be explained by large scale inhomogeneities, i.e. an LTB model.

- However, the past behaviour of LTB and ΛCDM models is different.



Suppose that we are in a $\Lambda = 0$, LTB universe. In the past, 6 Gyrs ago, could the observational data also have been interpreted as ΛCDM?





No: If the universe is LTB, then not only are we at a special place, but also at a special time

Going beyond the point of reconvergence on the past null cone

- Due to the expansion of the universe, the past null cone has a maximum size (where $r = r_{\mathrm{max}}$), and beyond this it reconverges. In EdS, r_{max} is at $z = 1.25$.

- At $r = r_{\mathrm{max}}$ the coordinates are singular, and the code can be used only in a domain $r < r_{\mathrm{max}} - \epsilon$.

- This is a coordinate problem and is resolved by using new coordinates.

Affine radial coordinate

From the geodesic equation

$$\frac{dr}{dv} = e^{-2\beta}. \tag{1}$$

Apply the tensor transformation law to the Bondi-Sachs metric, and re-write it in the form

$$ds^2 = -\left(1 + \frac{W}{D}\right) du^2 - 2du dv + D^2\{d\theta^2 + \sin^2\theta d\varphi^2\} \tag{2}$$

with: $W = W(u, v)$ and $r \to D = D(u, v)$.

Substitution into the Einstein field equations gives

$$D_{,vv} = -\frac{1}{2}\kappa D\rho(v_1)^2 \tag{3}$$

$$D_{,uv} = \frac{1}{2}\Big\{W_{,v}D_{,v} + DD_{,vv} + WD_{,vv} - 2D_{,u}D_{,v} - 1 \tag{4}$$

$$+ (D_{,v})^2 + \frac{1}{2}\kappa\rho D^2 - vD^2\Big\}\Big/ D \tag{5}$$

$$W_{,vv} = \frac{W}{D}D_{,vv} + 4D_{,uv} + 2\kappa\left(v_0v_1\rho - \frac{1}{2}\rho\right)D + 2vD \tag{6}$$

with: $D(0) = W(0) = W_{,v}(0) = D_{,u}(0) = 0$ and $D_{,v}(0) = 1.$

Then substituting the dust stress-tensor ($T_{ab} = \rho v_a v_b$) into the conservation equation, $T^{ab}_{\ ;b} = 0$, yields the fluid equations

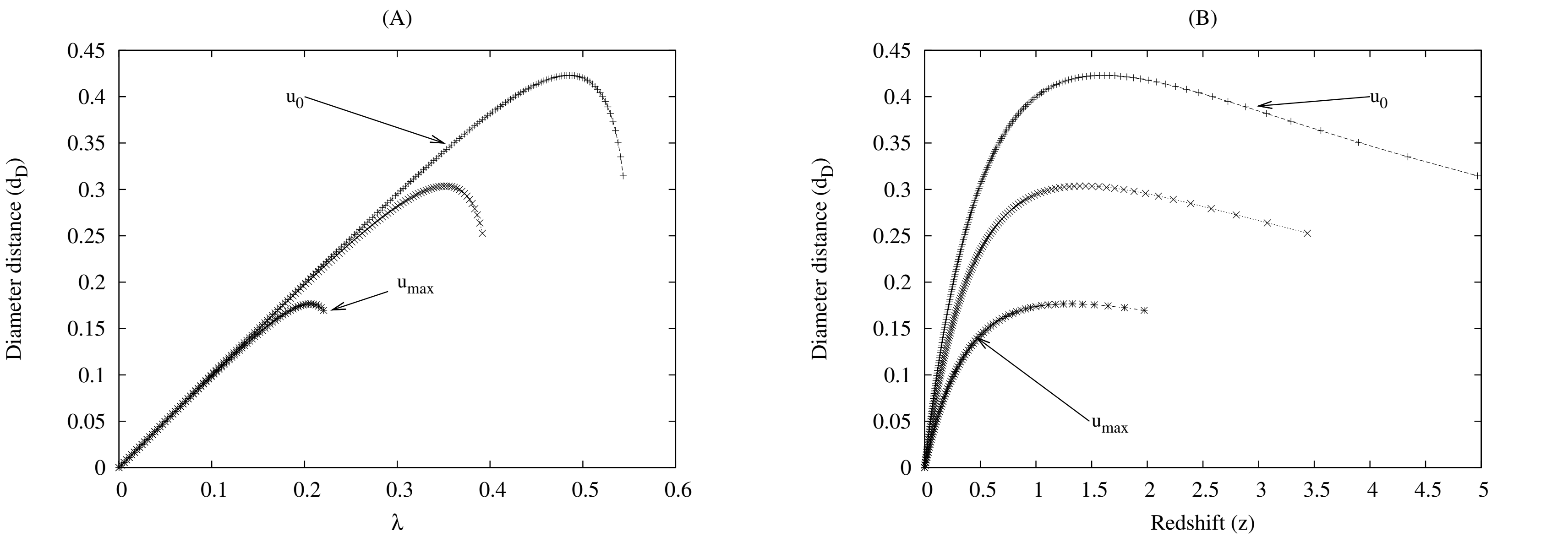
$$v_{1,u} = \frac{1}{v_1}\left\{\left(\left(1+\frac{W}{D}\right)v_1 - v_0\right)v_{1,v} + \frac{1}{2}(v_1)^2\left(\frac{W_{,v}}{D} - \frac{D_{,v}W}{D^2}\right)\right\} \tag{7}$$

$$\begin{aligned}\rho_{,u} = \frac{1}{v_1}\Bigg\{\rho\Bigg[&\left(1+\frac{W}{D}\right)\left(\frac{2v_1}{D}D_{,v} + v_{1,v}\right) - \left(\frac{2v_0}{D}D_{,v} + v_{0,v}\right) - \left(\frac{2D_{,u}}{D}\right)v_1 \\ &+ \left(\frac{W_{,v}}{D} - \frac{D_{,v}W}{D^2}\right)v_1\Bigg] + \rho_{,v}\left(\left(1+\frac{W}{D}\right)v_1 - v_0\right) - \rho v_{1,u}\Bigg\}\end{aligned} \tag{8}$$

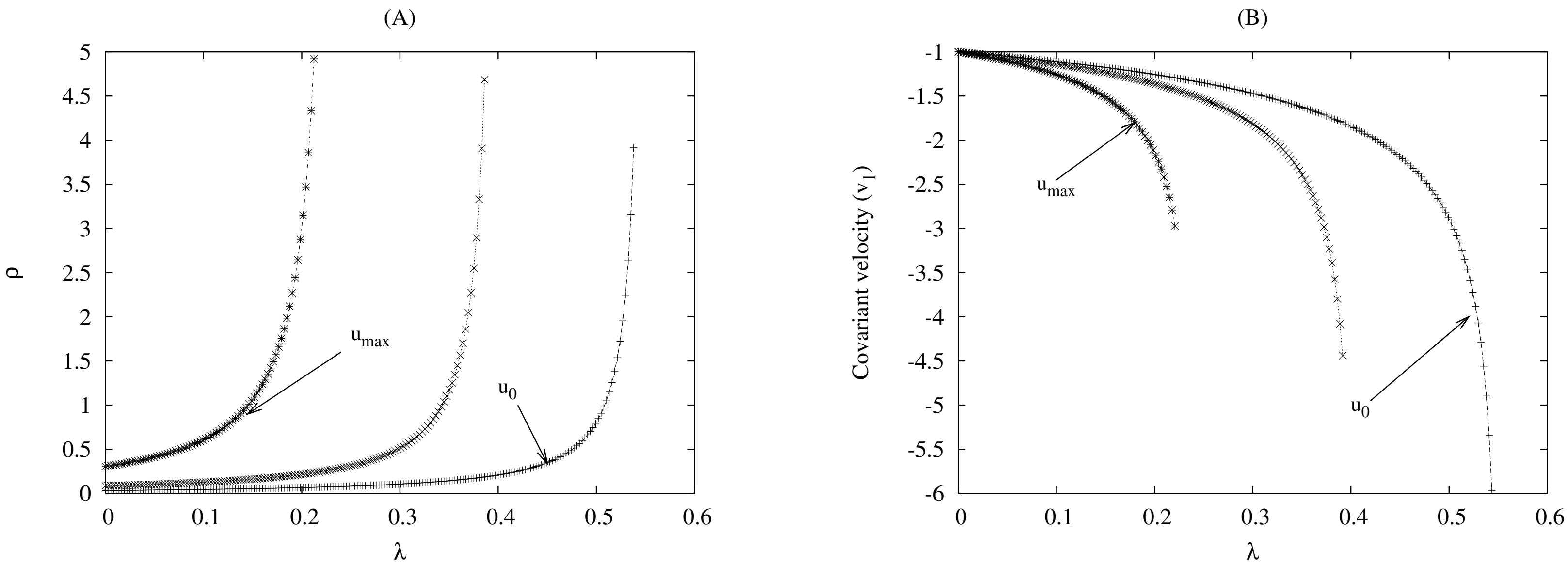
Here $v_1 = -(1+z)$, and using the condition $g^{ab}v_a v_b = -1$, v_0 can be written as

$$v_0 = \frac{1}{2}\left(1+\frac{W}{D}\right)v_1 + \frac{1}{2}v_1^{-1}. \tag{9}$$

ΛCDM with $\Omega_\Lambda = 0.7$



Diameter distance against v (A) and against of z (B) on PNCs at different proper times (u) evolved from a local PNC up to $z = 5$.



Density distribution (A) and covariant velocity (B) on PNCs at different proper times (u) evolved from a local PNC up to $z = 5$.

Using real data

Much data is processed assuming the FRW model, and the availability of data that is independent of cosmological model is limited. We use

- Union 2.1 SnIa data: $D(z)$
- Cosmic chronometers: dz/dt, where t measures proper time along the worldline of a galaxy

- Derived quantities:

$$H(z) = -\frac{1}{1+z}\frac{dz}{dt}$$
$$v(z) = \int_0^z \frac{1}{(1+z)^2 H}\, dz \to z(v)$$
$$D(z) \to D(v)$$
$$\rho = -\frac{2\partial_v^2 D}{\kappa D(1+z)^2}$$

- Cosmological constant Λ: Either assume a value, or regard Λ as a random variable uniformly distributed over $[0, \Lambda_{max}]$ where Λ_{max} is an upper bound on Λ such that $\Omega_\Lambda = 1$.

Monte Carlo method

Each element in a set of experimental data such as $(D_i, z_i), i = 1, \cdots, n$ is not known precisely but has a standard deviation σ_i attached to it. It is straightforward to construct a random variable X_{Di}, with mean D_i and standard deviation σ_i, to represent the result of a measurement of D at z_i. Here we need a to construct a random process to draw a realization of $D(z)$ that is a smooth function with values at nearby points highly correlated. We use Gaussian Process Regression as implemented in GaPP to do so.[§]

[§]M Seikel *et al.* JCAP **06** 036 (2014)

Algorithm 1

- Draw realizations of $D(z), dz/dt(z)$

- Compute derived quantities $D(v), \rho(v), v_1(v)$ and run characteristic code

- Compute desired quantities on and inside past null cone; discard realization if unphysical $-\ \rho < 0$ anywhere, age $t^* <$ 10.5Gyr, or shell-crossing.

Algorithm 2

- Draw 1000 realizations, and check that they are all physical

- Make a (small) random step within the function space, and perform algorithm 1

- Determine probabilities of old and new realizations fitting the data, and use this in a weighted probability formula to decide whether or not to accept the step

- For the 1000 realizations, determine the mean, 1σ, 2σ values for derived qunatities

- Take another random step, and repeat until mean, 1σ, 2σ values do not change significantly

- Double the number of realizations, and repeat all the above, and see if there is any significant change to the mean, 1σ, 2σ values. If yes, double the realizations again and repeat.

Testing the Copernican principle

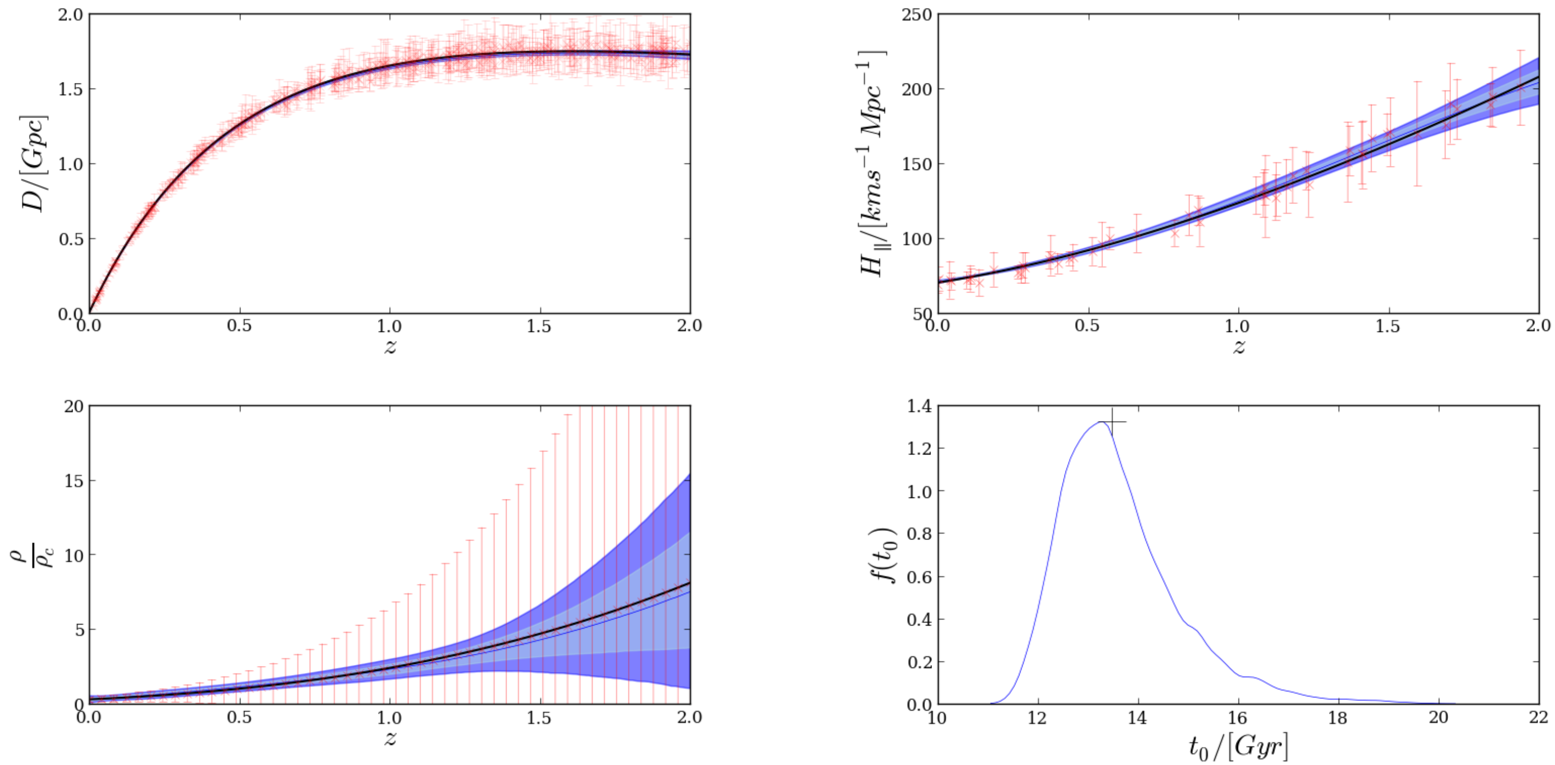
We construct quantities T_1, T_2 that are zero in a homogeneous universe (FRW), but non-zero otherwise, and see if the data can determine whether or not $T_1, T_2 = 0$.

- T_1 measures the difference between the radial (H) and tangential ($H_\perp$) expansion rates,

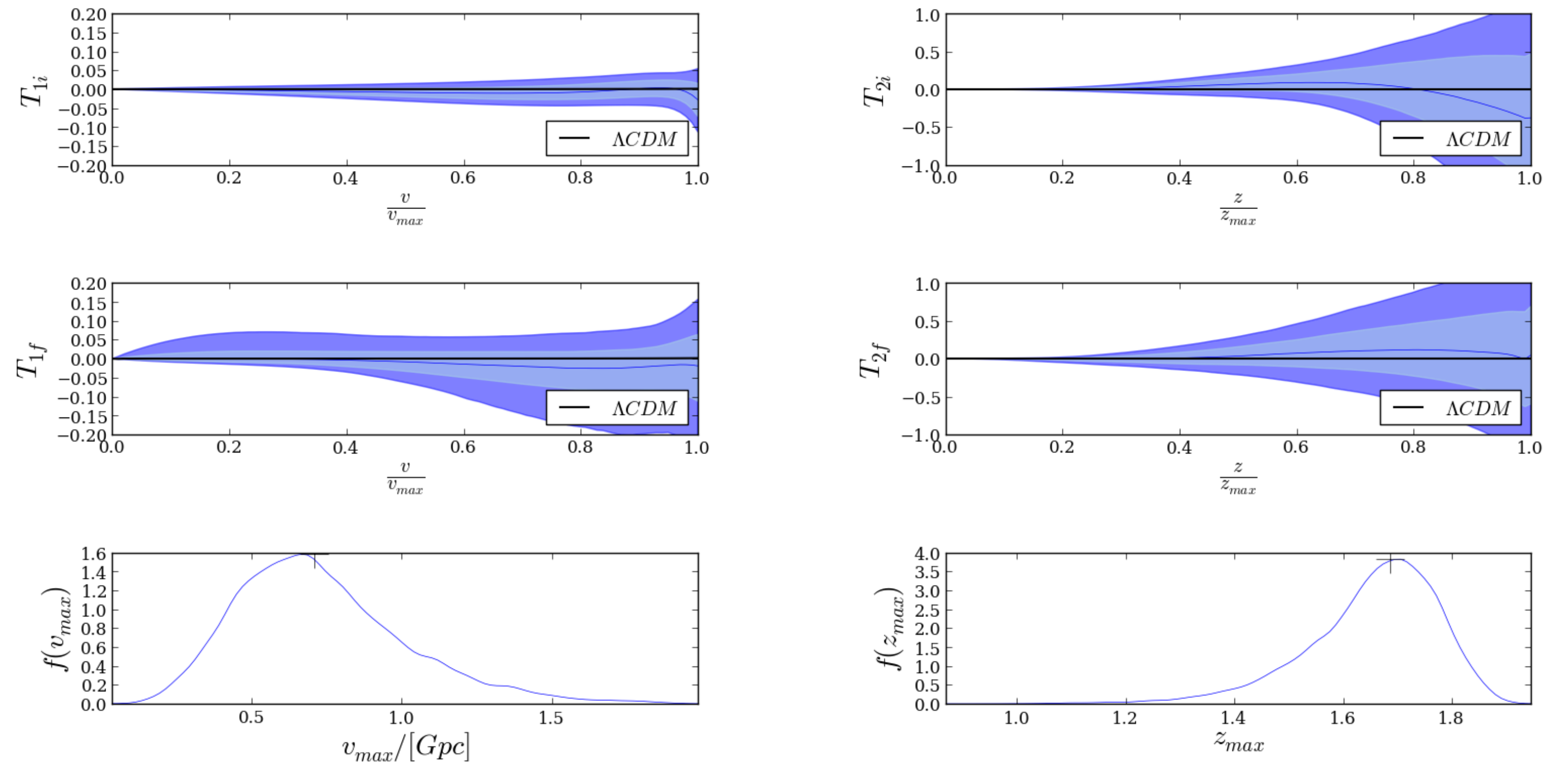
$$T_1 = 1 - \frac{H_\perp}{H} = 1 - \frac{1}{HD}\left((1+z)\partial_u D - \frac{\partial_v D}{2(1+z)} + \frac{(1+z)W\partial_v D}{2}\right)$$

- In FRW the curvature constant $k = \pm 1, 0$ and does not vary. This can be used to construct

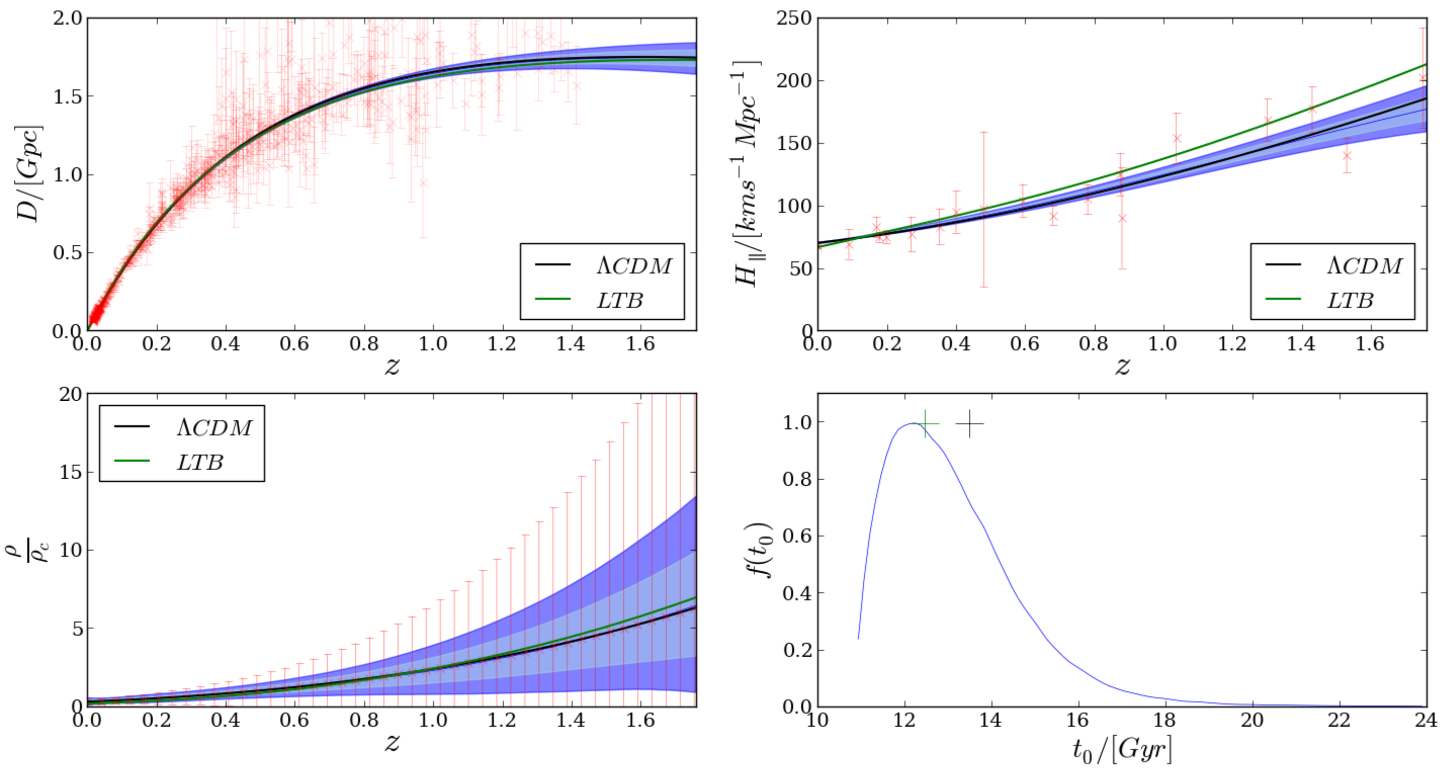
$$\begin{aligned} T_2 = 1 + H^2[(1+z)^2(D\partial_z^2 D - (\partial_z D)^2) - D^2] \\ +(1+z)HD\partial_z H[(1+z)\partial_z D + D]\,. \end{aligned}$$



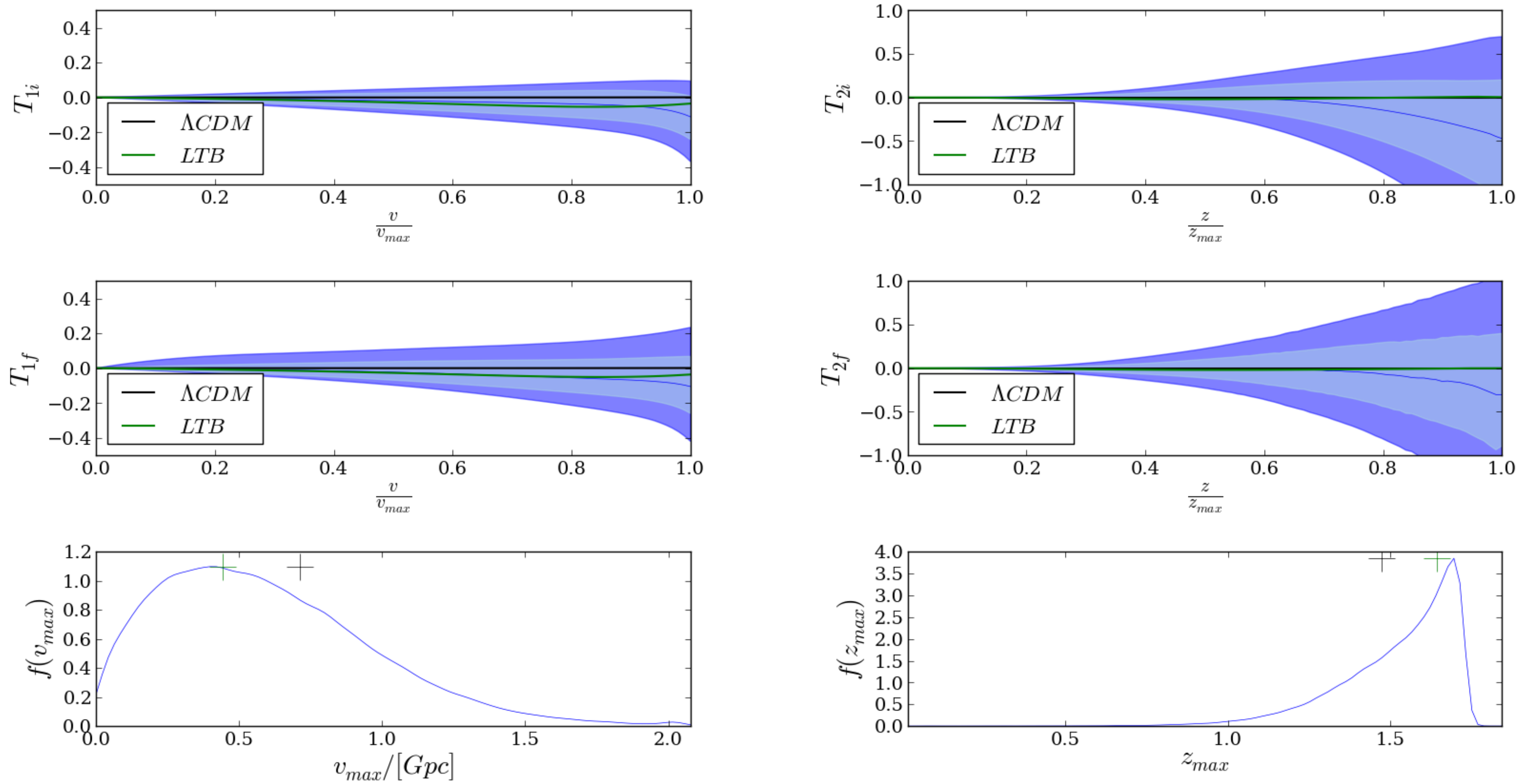
Simulated ΛCDM data



Simulated ΛCDM data



Real data



T_{1i}
T_{2i}
T_{1f}
T_{2f}
ΛCDM
LTB
$\frac{v}{v_{max}}$
$\frac{z}{z_{max}}$
$f(v_{max})$
$v_{max}/[Gpc]$
$f(z_{max})$
z_{max}

Real data

Discussion

- Simulated data tests behave as expected
- LTB model has $\Lambda = 0$ with parameters set to fit SnIa data
- Expect quality of data sets to improve with time
- We are investigating how to use data sets, such as galactic number counts, which are not independent of the cosmological model

- For the future, a direct measurement of red-shift drift would be important. With 3 independent data sets, we can determine both the geometry of the past null cone and the theory of gravity on cosmological scales: The value of Λ could be determined, and even whether or not it is distance-dependent.

Cosmology with 3D characteristic code

- Characteristic extraction code has been adapted for cosmology, i.e. inclusion of matter as dust, and with a central region excluded. The code is currently being tested.

- Little prospect of data for V (proper motion) or J (gravitational waves).

- Instead, will be used to explore What if? questions concerning cosmological averaging: strongly inhomogeneous density, importance of angular motions, gravitational waves, etc.

Conclusions

- The characteristic formalism can be used for numerical evolutions in vacuum, or with matter.

- The formalism is use for the extraction of gravitational radiation from a "3+1" simulation.

- The characteristic code can be used, in principle, to compute the past behaviour of the universe from observations.

- If the universe is $\Lambda = 0$ LTB rather than ΛCDM, then not only are we in a special position, but we are also at a special time.

- In order to get past the reconvergence of the past null cone, the code has been reformulated using an affine, rather than a surface area, radial parameter.

- The code can be used with real data to constrain the cosmological model, although at present models are not strongly constrained.

- Further sharpening of data, and additional data sets, should lead to interesting results.

- The 3D characteristic cosmological code will be able to explore the importance of averaging in cosmology.

THANK YOU