Advanced General Relativity: Exercises

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Listed below are the exercises that have been assigned during the course and collected according to the lecture in which they were assigned. These exercises can be solved independently or together during the exercise time. Some of these questions could be part of the oral exam.

Lecture I

1. Prove the following relation holds for the Riemann tensor
$$3R_{\alpha[\beta\gamma\delta]} = R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta}\,. \tag{1}$$
Next, using the Bianchi identities
$$\nabla_{[\alpha}R_{\beta\delta]\mu\nu} = 0 = \nabla_{\alpha]}R_{\mu\nu[\beta\delta}\,, \tag{2}$$
show there exists a tensor
$$G^{\alpha\beta} \coloneqq R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\,, \tag{3}$$
where $R_{\alpha\beta} \coloneqq R^{\mu}_{\ \alpha\mu\beta}$ and $R \coloneqq R^{\alpha}_{\ \alpha}$ are respectively the Ricci tensor and Ricci scalar and such that
$$\nabla_{\alpha}G^{\alpha\beta} = 0\,. \tag{4}$$

2. Prove that if $\boldsymbol{\xi}$ is the separation four-vector between two neighbouring geodesics with tangent vector $\boldsymbol{u}$, then the following expression can be derived
$$\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{u}}\xi^{\alpha} = -R^{\alpha}_{\ \beta\mu\nu}u^{\beta}\xi^{\mu}u^{\nu}\,. \tag{5}$$
This is known as the geodesic-deviation equation.

3. ***Optional.*** Consider the Einstein equations written generically as a tensor expression equating geometry and energy, *i.e.*,
$$G_{\alpha\beta} + \kappa_1\Lambda g_{\alpha\beta} = \kappa_2 T_{\alpha\beta}\,, \tag{6}$$
where κ_1 and κ_2 are two generic constant and Λ the cosmological constant. Show that the matching with the Newtonian limit yields
$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}\,. \tag{7}$$

Lecture II

1. Show that when using the Lagrangian
$$2L = \mathcal{L}^2 = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\,, \tag{8}$$
where $\dot{x}^{\mu} = dx^{\mu}/d\lambda$ and λ is the affine parameter of a massive particle, the *Euler–Lagrange equations*
$$\frac{\partial L}{\partial x^{\mu}} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right) = 0\,, \tag{9}$$
yield the geodesic equations
$$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\ \mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0\,. \tag{10}$$

2. Using as affine parameter $\lambda = \tau/m$, where m is the mass of the particle, show that the Lagrangian (8) relative to a Schwarzschild spacetime and the Euler–Lagrange equations yield the following geodesic equations

$$\frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}\right] = 0\,, \tag{11}$$

$$\begin{aligned}\frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)^{-1}\frac{dr}{d\tau}\right] = &\ r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right] \\ &- \frac{M}{r^2}\left[\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-2}\left(\frac{dr}{d\tau}\right)^2\right]\,,\end{aligned} \tag{12}$$

$$\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right) = r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2\,, \tag{13}$$

$$\frac{d}{d\tau}\left(r^2\sin^2\theta\frac{d\phi}{d\tau}\right) = 0\,. \tag{14}$$

3. Show that the constant of motions given by the covariant components of a particle's four-momentum p_t and p_ϕ represent the energy at infinity and the specific angular momentum, respectively.

4. ***Optional.*** Show that the lowest specific angular momentum allowing for the existence of circular orbits is $\tilde{\ell}^2 := (\ell/m)^2 = 12M^2$. Calculate the radii of the corresponding stable and unstable circular orbits.

Lecture III

1. Show that when considering the Lagrangian

$$2L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\,, \tag{15}$$

for a massive particle of mass m, the following identity holds

$$L = -\frac{1}{2}m^2\,. \tag{16}$$

2. Derive the values of the specific angular momentum and specific energy for a massive particle in circular orbit around a Schwarzschild spacetime.

3. Derive the values of the specific angular momentum relative to the marginally bound orbit for a massive particle in a Schwarzschild black hole.

4. ***Optional.*** Calculate the expression of the tetrad components carried by a Zero Angular Momentum Observer (ZAMO).

Lecture IV

1. A particle with rest mass m and four-momentum $\boldsymbol{p} = m\boldsymbol{v}$ is analysed by an observer with four-velocity $\boldsymbol{u}$.

   - Compute the total energy E of the particle.
   - Compute the kinetic energy E_T of the particle.
   - Compute the magnitude of the spatial momentum $p := \sqrt{p^i p_i}$.
   - Compute the magnitude of the three velocity $v := \sqrt{v^i v_i}$.

2. Define the four-acceleration of a particle with four-velocity $\boldsymbol{u}$ as

$$a^\mu := \frac{du^\mu}{d\tau}\,, \tag{17}$$

   where τ is the proper time. Show that $\boldsymbol{a} \cdot \boldsymbol{u} = 0$, *i.e.*, the acceleration is orthogonal to the four-velocity. What does this mean in a frame comoving with the particle?

3. Consider a photon emitted by a static observer in a Schwarzschild spacetime and and propagating in the direction $\boldsymbol{k}$. Let ψ be the angle between the direction of propagation of the photon and the unit radial four-vector of the tetrad carried by the static observer. Compute at what angles an *ingoing photon* should be fired to reach infinity if the observer is at $r = 6M, r = 3M$ and $r = 2M$. Repeat the considerations for an *outgoing photon*. Draw a sketch to illustrate the behaviour.

Lecture V

1. Show that the frame-dragging angular in a Kerr spacetime is given by the expression

$$\omega(r,\theta) := \frac{p^\phi}{p^t} = \frac{2Mra}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}\,. \tag{18}$$

   Further show that the angular velocity of a massive particle is constrained to be

$$\Omega_{\rm min} \le \Omega \le \Omega_{\rm max}\,, \tag{19}$$

   where

$$\Omega_{\rm min,max} := \omega \mp \sqrt{\omega^2 - g_{00}/g_{\phi\phi}}\,. \tag{20}$$

   Plot these two angular velocities for different values of the spin parameter. What is the angular velocity of the horizon?

2. Show that the specific energy and specific angular momentum for circular orbits of massive particles in a Kerr spacetime are given by

$$\tilde{E} = \frac{r^2 - 2Mr \pm a\sqrt{Mr}}{r\left(r^2 - 3Mr \pm 2a\sqrt{Mr}\right)^{1/2}}\,,$$

$$\tilde{\ell} = \pm\frac{\sqrt{Mr}\left(r^2 \mp 2a\sqrt{Mr} + a^2\right)}{r\left(r^2 - 3Mr \pm 2a\sqrt{Mr}\right)^{1/2}}\,,$$

where the $\pm$ signs refers to corotating and counterrotating particles, respectively.

3. ***Optional.*** Compute the expression for the Keplerian angular velocity for a massive particle in a Kerr spacetime and the corresponding specific angular momentum.