Consistent models for the structure of strongly magnetized neutron stars

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**MOTIVATION**

- high B neutron stars: XDINSs and RRATs, SXP, ESP, having super critical magnetic fields
- Soft Gamma-ray Repeaters (SGRs), Anomalous X-ray Pulsars (AXPs)
- Observations indicate common features \( \Rightarrow \) SGRs/AXPs belong to a unified class of objects, i.e. magnetars

\[
\begin{align*}
\text{M. Burgay (2009)} & \\
\text{Ho, Klus, Coe, Andersson (2013)} & \\
\end{align*}
\]
**MAGNETARS**

- **quiescence**: all SGRs/AXPs display a steady luminous X-ray emission with emission $L_x \sim 10^{35}$ erg/s

- characterized by X-ray bursting, flaring, and outbursts, Luminosities super-Eddington $\sim 10^{46}$ erg s$^{-1}$

- Large luminosities at quiescence and during outbursts can be explained in terms of magnetic dissipation

- long periods ($P \sim 5$-12 s), large spin-down rates ($P_{\text{dot}} \sim 5 \times (10^{-13} - 10^{-10})$ s/s)

- $dE/dt > dE_{\text{rot}}/dt \Rightarrow$ powered by field decay

- $P$-$P_{\text{dot}}$, magnetic braking $\Rightarrow B \sim 10^{14} - 10^{15}$ G
  (Duncan & Thomson 1992; Thomson & Duncan 1993)

- Direct measurements of the field (Ibrahim et al.)
**MAXIMUM ALLOWED MAGNETIC FIELD**

- **Inside the star, the magnetic field may be even higher**
- **The limiting interior field strength for a star can be estimated using the Virial theorem,**
  \[ 2T + W + 3 \Omega + M = 0 \]
  - \( T = \) total rotational kinetic energy
  - \( W = \) gravitational potential energy
  - \( \Omega = \) internal energy
  - \( M = \) magnetic energy

  Since \( T, \Omega > 0 \), maximum magnetic energy can be comparable to, but cannot exceed gravitational energy in equilibrium

- **For a typical neutron star, \( B_{\text{max}} \approx 10^{18} \text{ G} \)**
NEUTRON STAR STRUCTURE
Equation of state (EoS)

Tolman-Oppenheimer-Volkov equations of relativistic hydrostatic equilibrium:

\[
\begin{align*}
\frac{dp}{dr} &= -\frac{G (m + 4\pi pr^3)(\epsilon + p)}{c^2 r (r - 2Gm/c^2)} \\
\frac{dm}{dr} &= 4\pi \frac{\epsilon}{c^2} r^2
\end{align*}
\]
Constraining the EoS

$M^\text{max}(\text{theo}) > M^\text{max}(\text{obs})$

Demorest et al (Nature 2010)

\[ 1.97 \pm 0.04 \, M_\odot \]

Antoniadis et al (Science 2013)

\[ 2.01 \pm 0.04 \, M_\odot \]

Lattimer and Prakash, arXiv:1012.3208

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PARTICLES IN MAGNETIC FIELD: LANDAU QUANTIZATION

- In the presence of the magnetic field, the motion of the charged particles is Landau quantized in the direction perpendicular to the magnetic field.
- The critical field (the value where the cyclotron quantum is equal to the rest energy of the charged particle) for electrons is $B_m^{(e)(c)} = 4.4 \times 10^{13} G$, and for protons it is $B_m^{(p)(c)} \sim 10^{20} G$
- Choosing the coordinate axes such that $B$ is along $z$-axis, the single particle energy of any charged particle at $n$-th Landau level is given by

$$E_n = \sqrt{p_z^2 + m^2 + 2ne|Q|B}.$$
Magnetic field effects in Neutron Stars


effects of magnetic field on the dense matter
Equation of State

interaction of the electromagnetic field with matter (magnetisation)

anisotropy of the energy momentum tensor caused by breaking of the spherical symmetry by the electromagnetic field
to calculate the structure and observable properties of the neutron star within General Relativistic framework
ε coupling scheme, the TOV relations \[7\] allow us to describe a magnetic field, where a dashed and dot interaction: 

\[
\rho_\text{mag} = 0.5 \times 10^5 G, \quad \mu_\text{mag} = 7 \times 10^8 G
\]

at very high baryon chemical potential, which we vary. The reference in the calculation for our model is that small 

\[
\frac{P}{B} \leq 10 \quad \text{and} \quad \frac{f}{c} \leq 10^{-5}
\]

In addition, we would like to point out that our results are not qualitative. It is important to note that we assume that the magnetic field is conserved, and that this parameter, \(B\), is a self-consistent gap parameter (not a constant), which in turn forces the construction of axisymmetric models within the isotropic TOV formalism only for the interior of the star. This conclusion is based on the Gibbs construction of the mixed phase. 

The properties of the stars are strongly influenced by the magnitude of the magnetic field, which give rise to larger magnetic fields in the low-density layers. The change in the order that the hyperons appear as a function of magnetic field at the centre of a compact star may be obtained. 

The full line indicates the M-q line. The dense part of the figure is not shown as it exhibits a self-consistent gap parameter (not a constant), which in turn forces the construction of axisymmetric models within the isotropic TOV formalism only for the interior of the star. This conclusion is based on the Gibbs construction of the mixed phase. 

The mass-radius relation for the EoS given in \[8\] for CFL hyperon star models, and other results are presented without and with (variable) magnetic field. The properties of the stars are strongly influenced by the magnitude of the magnetic field which in turn forces the construction of axisymmetric models within the isotropic TOV formalism only for the interior of the star. This conclusion is based on the Gibbs construction of the mixed phase.
Magnetic field effects in Neutron Stars

- Effects of magnetic field on the dense matter
  - Equation of State

- Interaction of the electromagnetic field with matter (magnetisation)

- Anisotropy of the energy momentum tensor caused by breaking of the spherical symmetry by the electromagnetic field

To calculate the structure and observable properties of the neutron star within General Relativistic framework

Bonazzolla, Gourgoulhon, Salgado, Marck (1993)
Cardall, Prakash, Lattimer (2001)

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AIM OF THE STUDY

Consistent neutron star models in a strong magnetic field

- effects of magnetic field on the dense matter
  Equation of State

- interaction of the electromagnetic field with matter (magnetisation)

- anisotropy of the energy momentum tensor caused by breaking of the spherical symmetry by the electromagnetic field

- to calculate the structure and observable properties of the neutron star within General Relativistic framework
**MICROSCOPIC ENERGY MOMENTUM TENSOR**

- Lagrangian density of a fermion system in the presence of a magnetic field
  \[
  \mathcal{L} = \bar{\psi}(x)(D_\mu \gamma^\mu - m)\psi(x) - \frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}
  \]

  where \( D_\mu = i\partial_\mu - eA_\mu \)

  and the field strength tensor of the electromagnetic field is
  \[
  F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
  \]

- In flat space, the Canonical energy momentum tensor
  \[
  \Theta^{\mu\nu} = \sum_{\varphi} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \partial_\nu \varphi - g^{\mu\nu} \mathcal{L}
  \]

  is the conserved Noether current associated with the symmetry of space-time translations.

- This is neither symmetric nor gauge-invariant, hence unsuitable as source of Einstein equations.

- The energy-momentum tensor can be written in a symmetrized and gauge invariant form (Belinfante-Rosenfeld tensor)

  L. Rosenfeld, Acad. Roy. Belg., Memoires de classes de Science 18 (1940), F.J. Belinfante, Physica 7 (1940), 449.

  \[
  T^{\mu\nu} = \frac{1}{4\pi} \left( F^{\mu\alpha}F_\alpha^{\nu} + g^{\mu\nu} \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right) + \frac{1}{2} \bar{\psi}(\gamma^\mu D^\nu + \gamma^\nu D^\mu)\psi
  \]

  \[
  \text{field} \quad \text{matter}
  \]
THERMODYNAMIC AVERAGE OF ENERGY MOMENTUM TENSOR

- thermodynamic average of the microscopic energy-momentum tensor in the statistical ensemble

\[
\langle \tau^{\mu\nu} \rangle = \frac{1}{\beta V} \frac{1}{Z} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \int_0^\beta d\lambda \int d^3x \tau^{\mu\nu} \exp(\bar{S})
\]

where the partition function is

\[
Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp(\bar{S})
\]

- thermal average of the energy-momentum tensor is given by

\[
T_{m}^{\mu\nu} = \varepsilon_m u^\mu u^\nu - P(g^{\mu\nu} - u^\mu u^\nu) + \frac{1}{2} (M^{\mu\lambda} F_{\lambda}^{\nu} + M^{\nu\lambda} F_{\lambda}^{\mu})
\]

<table>
<thead>
<tr>
<th>pure fermionic</th>
<th>magnetisation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{m}^{\mu\nu}$ = $\varepsilon_m u^\mu u^\nu - P(g^{\mu\nu} - u^\mu u^\nu)$</td>
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<tr>
<th>field</th>
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</thead>
<tbody>
<tr>
<td>$T_{f}^{\mu\nu} = -\frac{1}{4\pi} F^{\mu\lambda} F_{\lambda}^{\nu} + \frac{1}{16\pi} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$.</td>
</tr>
</tbody>
</table>
In the rest frame of the hadronic fluid, with B field along the z axis, the matter and field parts of the energy-stress tensor are given by

\[
T^\mu_\nu = \begin{pmatrix}
\varepsilon_m & 0 & 0 & 0 \\
0 & P - MB & 0 & 0 \\
0 & 0 & P - MB & 0 \\
0 & 0 & 0 & P
\end{pmatrix},
\]

\[
T_f^\mu_\nu = \frac{B^2}{8\pi} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

In the literature, components of the energy-momentum tensor are defined as

total energy density \[ \varepsilon = \varepsilon_m + \frac{B^2}{8\pi}. \]
total “longitudinal” and “transverse” pressure

\[
P_{\parallel} = P - \frac{B^2}{8\pi},
\]

\[
P_{\perp} = P - MB + \frac{B^2}{8\pi}.
\]
**MAGNETIC FIELD DEPENDENT EQUATION OF STATE**

- Example: Quark Matter in MCFL phase  
  (Noronha and Shovkovy 2007, Paulucci et al. 2011)
- massless 3-flavor MIT Bag model (with \( B = 60 \text{ MeV/fm}^3 \)) + Pairing interaction of NJL-type

![Graph showing the effects of Landau quantization]

- effects of Landau quantization become noticeable only for fields of \( \sim 10^{19} \text{G} \).

![Graph showing magnetisation and de Haas-van Alphen oscillations]

- Magnetisation negligible
- de Haas-van Alphen oscillations


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The structure equations of neutron stars are obtained by solving Einstein’s field equations.

In the 3+1 Formalism, solving the Einstein’s equations (system of 2nd order PDEs) are reduced to integration of a system of coupled 1st order PDEs subject to certain conditions:

- 6 evolution equations for the extrinsic curvature
- 1 Hamiltonian constraint equation
- 3 momentum constraint equations

The formulation has been employed to construct a numerical code (LORENE) using spectral methods.

The code has been extended to include coupled Einstein-Maxwell equations describing rapidly rotating neutron stars with a magnetic field.

Bonazzolla, Gourgoulhon, Salgado, Marck (1993)

MAGNETOSTATIC EQUILIBRIUM
(WITHOUT MAGNETISATION)

• Equations for magnetostatic equilibrium (from the conservation of energy and momentum):
  \[ \nabla_{\mu} T^{\mu\nu} = 0 \]

• Inhomogeneous Maxwell equations:
  \[ \frac{1}{\mu_0} \nabla_{\mu} F^{\nu\mu} = j^{\nu}_{\text{free}} \]

  **Bonazzola, Gourgoulhon, Salgado, Marck (1993)**

• Einstein-Maxwell equations
  \[ \nabla_{\alpha} T^{\alpha\beta} = \nabla_{\alpha} T^{\alpha\beta}_f - F^{\beta\nu} j^{\nu}_{\text{free}} \]


• first integral of fluid stationary motion (momentum conservation):
  \[ \ln h(r, \theta) + \nu(r, \theta) - \ln \Gamma(r, \theta) + M(r, \theta) = \text{const.} \]

• In terms of enthalpy per baryon for neutron star
  \[ h := \frac{\varepsilon + p}{n_{b}} \]
  and
  \[ M(r, \theta) = - \int_{0}^{A(\varphi, r, \theta)} f(x) \, dx. \]

the electromagnetic term associated with the Lorentz force
MAGNETOSTATIC EQUILIBRIUM  
(WITH MAGNETISATION)

- In the Fluid Rest Frame, assuming perfect conductor, $E = 0$  
  $$F_{\mu\nu} = \epsilon_{\alpha\beta\mu\nu}u^\beta b^\alpha$$
- assuming isotropic medium, the magnetisation is aligned with the magnetic field  
  $$M_{\mu\nu} = \epsilon_{\alpha\beta\mu\nu}u^\beta m^\alpha$$
  $$m_\mu = \frac{x}{\mu_0} b_\mu$$

- Modified inhomogeneous Maxwell equations:
  $$\nabla_\mu F^{\sigma\mu} = \frac{1}{1 - x}(\mu_0 j_\text{free}^\sigma + [F^{\sigma\mu} \nabla_\mu x])$$  
  magnetisation

- first integral of fluid motion: 
  $$(\varepsilon + p) \left( \frac{1}{\varepsilon + p} \frac{\partial p}{\partial x^i} + \frac{\partial \mu}{\partial x^i} - \frac{\partial \ln \Gamma}{\partial x^i} \right) - F_{\mu\rho} j_{\text{free}}^\rho = 0$$

- it can be shown that 
  $$\frac{x}{2\mu_0} F_{\mu\nu} \nabla_i F^{\mu\nu} = \frac{x}{\mu_0} (b_\mu \nabla_i b^\mu - b_\mu b^\mu u_\nu \nabla_i u^\nu) = b \nabla_i b = \frac{m}{\varepsilon + p} \frac{\partial b}{\partial x^i},$$

- enthalpy per baryon for neutron star with magnetic field 
  $$h = h(n_b, b) = \frac{\varepsilon + p}{n_b}$$

- derivative of logarithm of enthalpy 
  $$\frac{\partial \ln h}{\partial x^i} = \frac{1}{\varepsilon + p} \left( \frac{\partial p}{\partial x^i} - m \frac{\partial b}{\partial x^i} \right)$$  
  Blandford and Hernquist (1982), Potekhin and Yakovlev (2012)
NUMERICAL RESOLUTION

• The structure equations of neutron stars are obtained by solving Einstein’s field equations
• In the 3+1 Formalism, solving the Einstein’s equations (system of 2nd order PDEs) are reduced to integration of a system of coupled 1st order PDEs subject to certain conditions:
  
  - 6 evolution equations for the extrinsic curvature
  - 1 Hamiltonian constraint equation
  - 3 momentum constraint equations

• The formulation has been employed to construct a numerical code (LORENE) using spectral methods

• The code has been extended to include coupled Einstein-Maxwell equations describing rapidly rotating neutron stars with a magnetic field

  Bonazzolla, Gourgoulhon, Salgado, Marck (1993)


• Incorporate magnetic field dependent EoS
• Incorporate magnetisation by modifying the inhomogeneous Maxwell equations
MAXIMAL DEFORMATION DUE TO MAGNETIC FIELD

Magnetic field lines and enthalpy isocontours in the meridional (x, z) plane for static configuration for $B_{\text{polar}} = 8.16 \times 10^{17} \text{ G}$, Mag moment = $3.25 \times 10^{32} \text{ Am}^2$, $M_G = 2.22 \, M_{\odot}$

- Stellar configurations strongly deviate from spherical symmetry
- Upon increasing magnetic field strength, the shape of the star becomes more and more elongated, finally reaching toroidal shape
**MAXIMUM GRAVITATIONAL MASS FOR A GIVEN MAGNETIC MOMENT**

- Gravitational mass varies with central log enthalpy and magnetic field
- Static configurations determined by different values of central log-enthalpy along constant sequences of magnetic dipole moment
- Plot of polar magnetic field corresponding to the values of magnetic moment for a neutron star of $M_B = 1.6 \, M_{\odot}$
- Maximum gravitational mass $M_G^{\text{max}}$ was determined by parabolic interpolation

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**EFFECT OF EoS(B) AND M**

- **The 3 cases:**
  1. Without magnetic field dependence in EoS, without magnetisation: no EoS(B), no M
  2. With magnetic field dependence in EoS, without magnetisation: EoS(B), no M
  3. With magnetic field dependence in EoS, with magnetisation: EoS(B), M

- **Maximal gravitational mass is an increasing function of magnetic moment**

- The effects of inclusion of magnetic field dependence of the EoS and the magnetisation are negligible, contrary to the claims of several previous works

• Observed magnetars are slowly rotating \((P \sim s)\)
• We chose a sequence of neutron stars rotating at 700 Hz, close to fastest known rotating pulsar (716 Hz)
• Maximum mass increases with magnetic moment
• Effect of magnetisation and magnetic field dependence of EoS again found to be negligible
The main effect arises from the purely electromagnetic part. Centrifugal forces exerted by the Lorentz force on matter increases with increasing magnetic moment. We studied the behaviour of compactness of a neutron star with baryon mass 1.6 with magnetic moment. This means that the fast rotating configurations do not have any realistic observed counterpart for the moment. The compactness was found to decrease with increase in magnetic moment, i.e. magnetic field, since the strong magnetic fields induce a very rapid spin-down. Finally, we computed rotating configurations along a family of curves of the order of seconds, mainly because the strong magnetic fields induce a very rapid spin-down. This should answer the question of the fate of accreting neutron stars.

We determined the structure of neutron stars in strong magnetic fields, relevant for the study of magnetars. Starting from the equation of state enter the final results. Equations of the electric field vanishes in the fluid rest frame, and therefor only magnetisation and the magnetic field dependence and magnetisation (see Fig.6).

The gravitational mass as function of central log-enthalpy shows in particular that, as claimed by Blandford & Hernquist, the main effect of the equation of state enter the final results. Equations of the electric field vanishes in the fluid rest frame, and therefor only magnetisation and the magnetic field dependence and magnetisation (see Fig.6).

We studied the behaviour of the compactness of a neutron star of baryon mass 1.6 with a gravitational mass of 2.0, along seven constant curves of magnetic dipole moment. The three models correspond to the possibility or not of inclusion of the magnetic field are very small, as in the Fig. (8) we see the same behaviour for both cases: the maximal mass increases with increasing magnetic moment, as illustrated in the Fig. (7). The compactness was found to increase with the magnetic dipole moment, i.e. magnetic field, since the strong magnetic fields induce a very rapid spin-down. This is understandable from the centrifugal forces exerted by the Lorentz force on matter increases with increasing magnetic moment.

We studied the behaviour of compactness of a neutron star with baryon mass 1.6 with magnetic moment.

**COMPACTNESS**

\[ C = \frac{GM_G}{R_{\text{circ}} c^2} \]

- We studied the behaviour of compactness of a neutron star with baryon mass 1.6 with magnetic moment.
- The compactness was found to decrease with increase in magnetic moment.
- Centrifugal forces exerted by the Lorentz force on matter increases with increasing magnetic moment.
- The influence of magnetic field dependence of EoS and magnetisation are negligible.
- The main effect arises from the purely electromagnetic part.

SUMMARY

• In this work, we developed a self-consistent approach to determine the structure of neutron stars in strong magnetic fields, relevant for the study of magnetars.

• Taking as an example the EoS of quark matter in MCFL phase, we investigated the effect of inclusion of magnetic field dependence of the EoS and magnetisation.

• In particular, it was found that the equilibrium only depends on the thermodynamic EoS and magnetisation explicitly only enters Einstein-Maxwell equations.

• In contrast to previous studies, we found that these effects do not significantly influence the stellar structure, even for the strongest magnetic fields considered.

• The difference arises due to the fact that in previous works isotropic TOV equations were used to solve for stellar structure, whereas magnetic field causes the star to deviate from spherical symmetry considerably.