The events GW150914 and GW151226: Gravitational waves from coalescing black-hole binaries

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Outline

- Details on GW150914 and GW151226
- Interferometric Detection of GWs
- Gravitational Waves
- Hamiltonian for General Relativity
- Binary Black-Hole Spacetimes
- Higher order post-Newtonian Hamiltonians
- Orbital Motion (ISCO)
- Spin-Gravity Interaction
- Data Analysis

1916, 1918 Einstein

1923 Eddington

1941 Landau/Lifshitz

1957 H. Bondi

1960, 1969 J. Weber, bar detector

1969 K. Thorne, radiation reaction

1972 R. Weiss (MIT), interferometric detector

1974 H. Billing (MPI), interferometric detector

1974, 1978 Hulse-Taylor pulsar, radiation damping

1989 LIGO (Caltech, MIT), Virgo (F-I), GEO600 (G-GB)

2002 LIGO, data aquisition

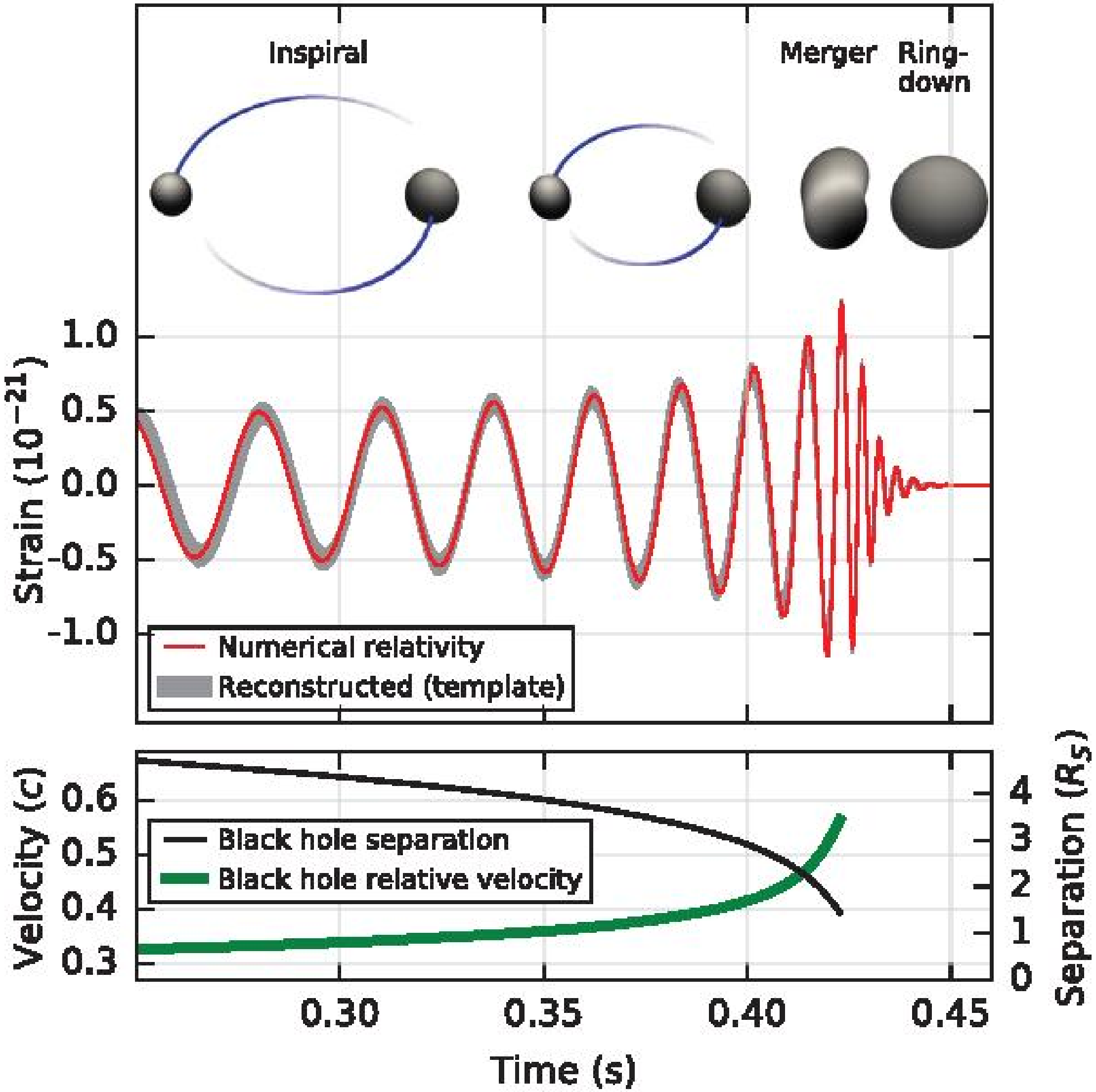
2015 aLIGO, data aquisition

Details on GW150914 and GW150226

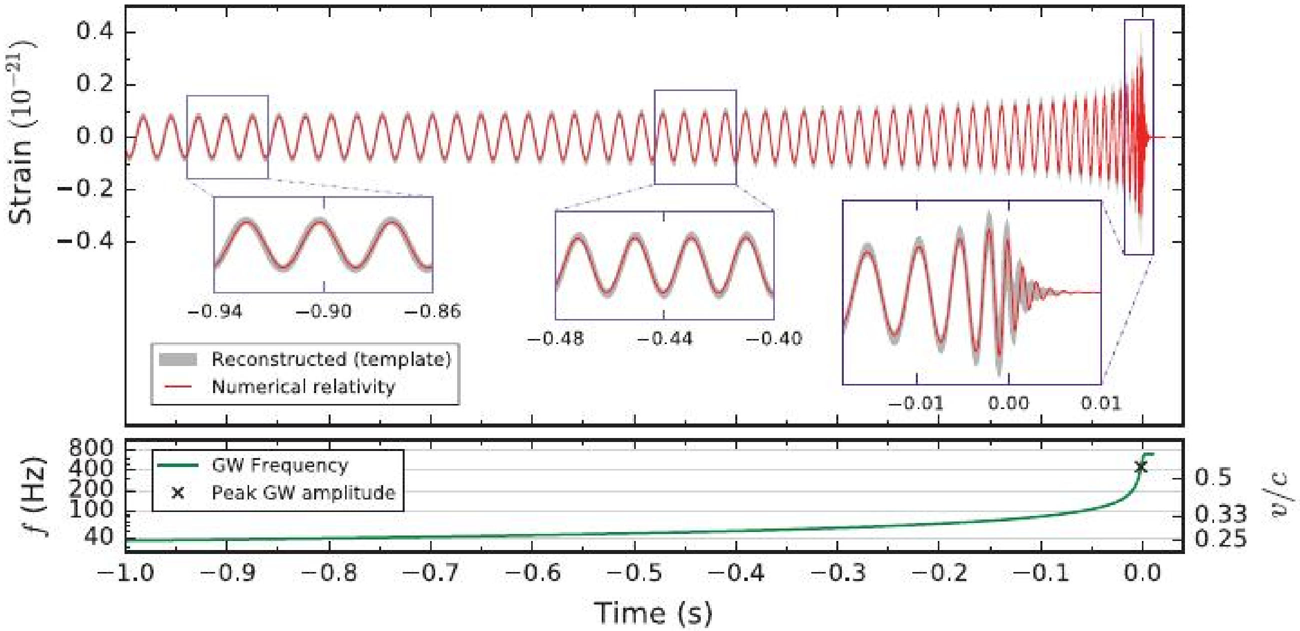
B. P. Abbott *et al.*, Phys. Rev. Lett. **116**, 061102 (2016)

B. P. Abbott *et al.*, Phys. Rev. Lett. **116**, 241103 (2016)

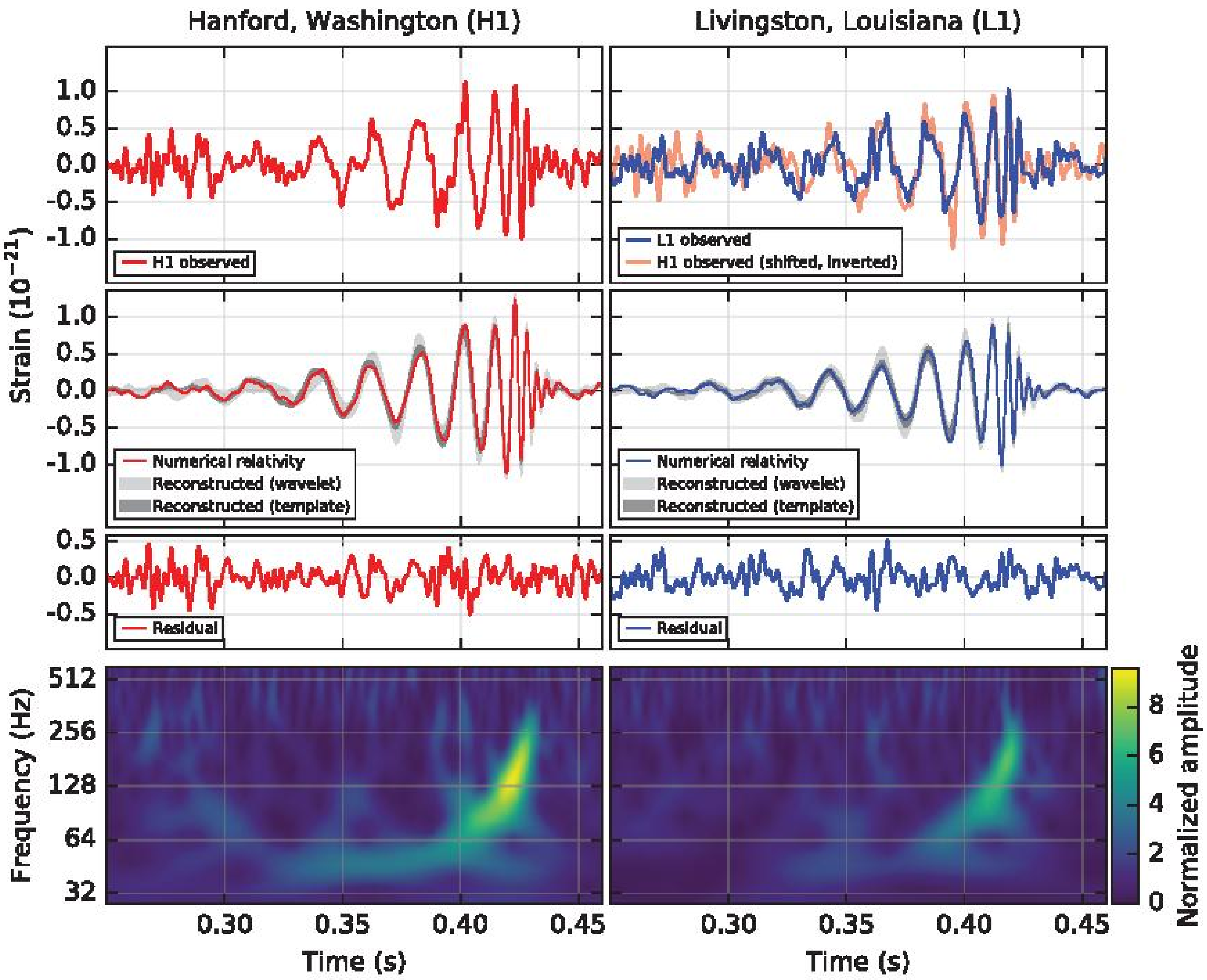
B. P. Abbott *et al.*, Phys. Rev. X **6**, 041015 (2016)



$$h(t)L = \Delta L(t) = \delta L_x - \delta L_y$$



Strain (10^{-21})
Reconstructed (template)
Numerical relativity
GW Frequency
Peak GW amplitude
f (Hz)
v/c
Time (s)



Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual
Frequency (Hz)
Normalized amplitude
Time (s)

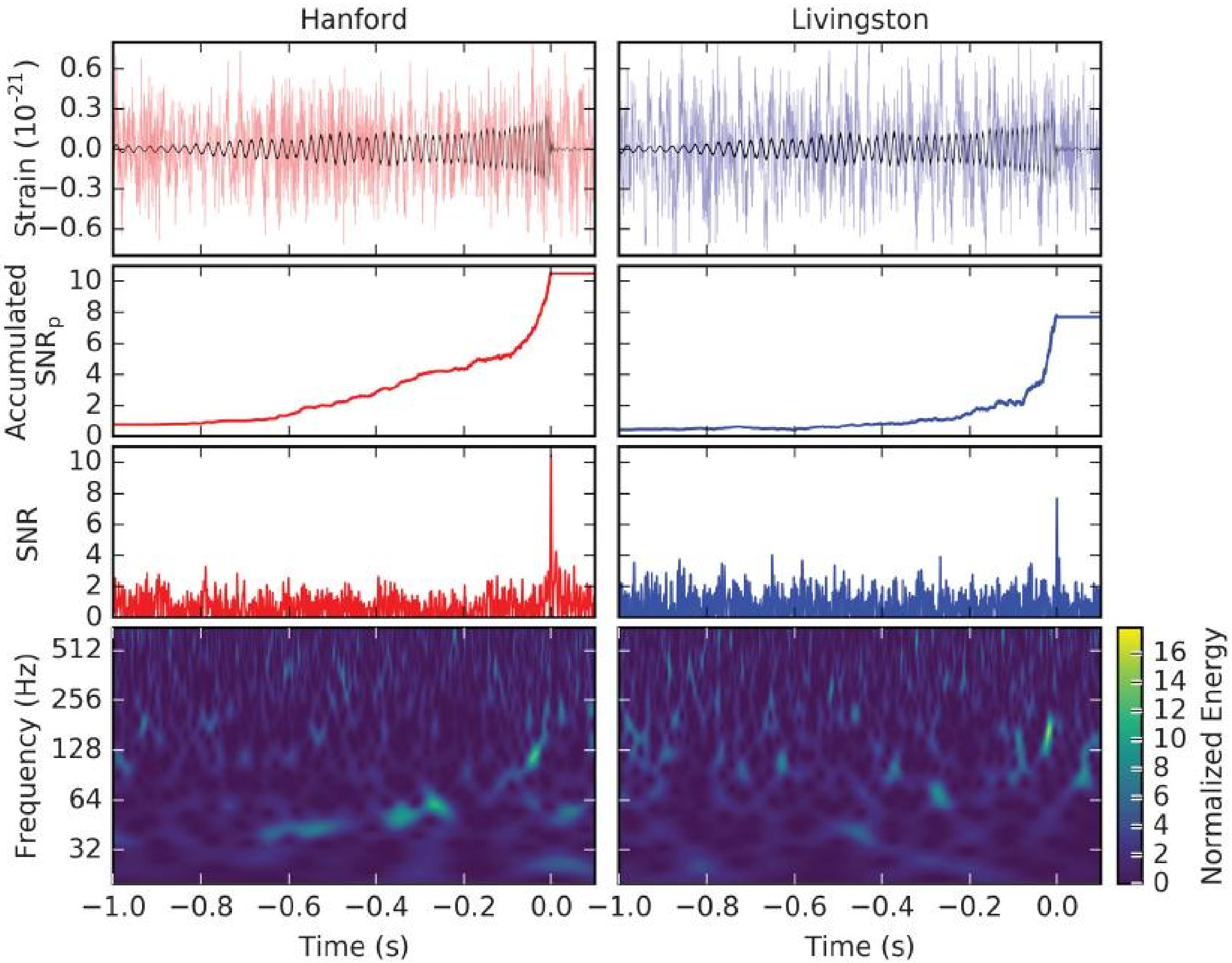

Hanford
Livingston
Strain (10^{-21})
Accumulated SNR_p
SNR
Frequency (Hz)
Normalized Energy
Time (s)
Time (s)

TABLE I. Source parameters for GW150914. We report median values with 90% credible intervals that include statistical errors, and systematic errors from averaging the results of different waveform models. Masses are given in the source frame; to convert to the detector frame multiply by $(1 + z)$ [90]. The source redshift assumes standard cosmology [91].

Primary black hole mass	$36^{+5}_{-4} M_\odot$
Secondary black hole mass	$29^{+4}_{-4} M_\odot$
Final black hole mass	$62^{+4}_{-4} M_\odot$
Final black hole spin	$0.67^{+0.05}_{-0.07}$
Luminosity distance	410^{+160}_{-180} Mpc
Source redshift z	$0.09^{+0.03}_{-0.04}$

TABLE I. Source parameters for GW151226. We report median values with 90% credible intervals that include statistical and systematic errors from averaging results of the precessing and nonprecessing spin waveform models. The errors also take into account calibration uncertainties. Masses are given in the source frame; to convert to the detector frame multiply by $(1+z)$ [61]. The spins of the primary and secondary black holes are constrained to be positive. The source redshift assumes standard cosmology [62]. Further parameters of GW151226 are discussed in [5].

Primary black hole mass	$14.2^{+8.3}_{-3.7} M_\odot$
Secondary black hole mass	$7.5^{+2.3}_{-2.3} M_\odot$
Chirp mass	$8.9^{+0.3}_{-0.3} M_\odot$
Total black hole mass	$21.8^{+5.9}_{-1.7} M_\odot$
Final black hole mass	$20.8^{+6.1}_{-1.7} M_\odot$
Radiated gravitational-wave energy	$1.0^{+0.1}_{-0.2} M_\odot c^2$
Peak luminosity	$3.3^{+0.8}_{-1.6} \times 10^{56}$ erg/s
Final black hole spin	$0.74^{+0.06}_{-0.06}$
Luminosity distance	440^{+180}_{-190} Mpc
Source redshift z	$0.09^{+0.03}_{-0.04}$

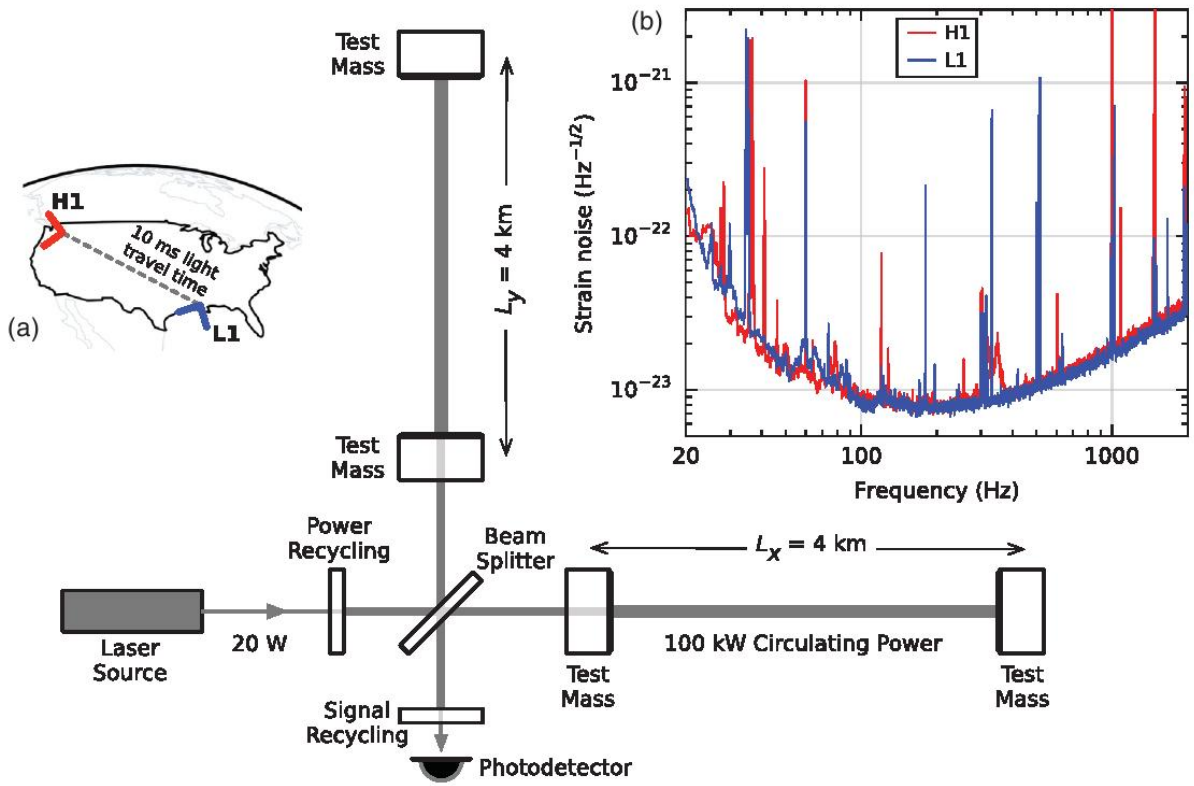
source power (3 solar masses radiated away in 0.2 seconds)

$$6 \times 10^{30}\text{kg}\ 9 \times 10^{16} \left(\frac{\text{m}}{\text{sec}}\right)^2 /200\text{ms} = 2.7 \times 10^{48}\text{Watts} = 0.75 \times 10^{-4}\frac{c^5}{G}$$

maximum power of a single process

$$\frac{c^5}{G} = \frac{Mc^2}{(GM/c^2)/c} = 3.6 \times 10^{52}\text{Watts}$$

Interferometric Detection of GWs



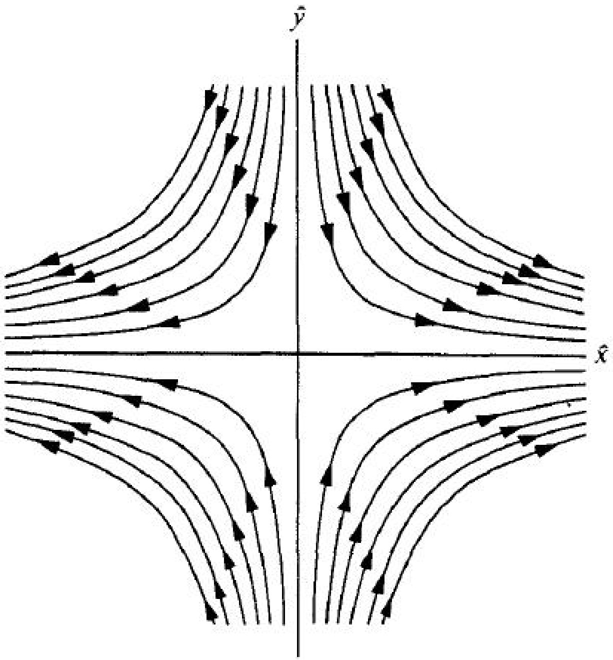
(a)
H1
L1
10 ms light travel time
(b)
H1
L1
Strain noise (Hz$^{-1/2}$)
10^{-21}
10^{-22}
10^{-23}
20
100
1000
Frequency (Hz)
Test Mass
Test Mass
L_y = 4 km
Power Recycling
Beam Splitter
L_x = 4 km
Laser Source
20 W
100 kW Circulating Power
Test Mass
Test Mass
Signal Recycling
Photodetector

gravitational wave $\quad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + h_{ij}^{\mathrm{TT}} dx^i dx^j$

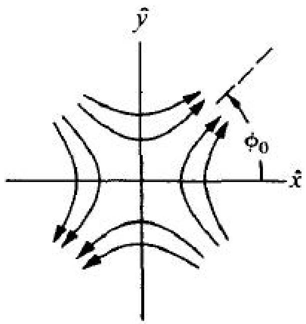
electromagnetic wave $\quad A = A_\mu dx^\mu = A_i^{\mathrm{T}} dx^i$

Fermi normal coordinates

$$ds^2 = -(1 - \frac{1}{2c^2}\ddot{h}_{ij}^{\mathrm{TT}} \hat{x}^i \hat{x}^j) c^2 d\hat{t}^2 + d\hat{x}^i d\hat{x}^i \qquad (|\hat{\mathbf{x}}| << c/f)$$



(a) Force lines for $\ddot{A}_\times = 0$, $\ddot{A}_+ > 0$



(b)

gravitational quadrupole wave

propagating in $\hat{z}$-direction

$$d^2\hat{x}/d\hat{t}^2 = \tfrac{1}{2}(\ddot{A}_+\hat{x} + \ddot{A}_\times\hat{y})$$

$$d^2\hat{y}/d\hat{t}^2 = \tfrac{1}{2}(-\ddot{A}_+\hat{y} + \ddot{A}_\times\hat{x})$$

$$d^2\hat{z}/d\hat{t}^2 = 0$$

MTW: Gravitation (1973)

chirp mass $$\mathcal{M} := \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} = \frac{c^3}{G}\left(\frac{5}{96}\frac{\dot{f}}{\pi^{8/3} f^{11/3}}\right)^{3/5}$$

velocity $$\frac{v}{c} = \left(\frac{GM\pi f}{c^3}\right)^{1/3}$$

gravitational quadrupole wave

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{2}{r}\frac{G}{c^4} P_{ijkm}(\mathbf{n}) \ddot{Q}_{km}\left(t - \frac{r}{c}\right)$$

$$Q_{km} = \sum_a m_a (x_k^a x_m^a - \frac{1}{3}\delta_{km} x_l^a x_l^a)$$

Gravitational Waves

Multipole expansion of far zone (FZ) field (e.g., Blanchet in LRR)

$$
\begin{aligned}
h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) &= \frac{G}{c^4}\frac{P_{ijkm}(\mathbf{n})}{r}\sum_{l=2}^{\infty}\left\{\left(\frac{1}{c^2}\right)^{\frac{l-2}{2}}\frac{4}{l!}\,\mathrm{M}^{(l)}_{kmi_3\ldots i_l}(t-\frac{r_*}{c})\;N_{i_3\ldots i_l}\right.\\
&+ \left.\left(\frac{1}{c^2}\right)^{\frac{l-1}{2}}\frac{8l}{(l+1)!}\,\epsilon_{pq(k}\;\mathrm{S}^{(l)}_{m)pi_3\ldots i_l}(t-\frac{r_*}{c})\;n_q\;N_{i_3\ldots i_l}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{M}_{ij}(t-\frac{r_*}{c}) &= \widehat{\mathrm{M}}_{ij}\left(t-\frac{r_*}{c}\right)\\
&+ \frac{2Gm}{c^3}\int_0^{\infty}dv\,\ln\left(\frac{v}{2b}\right)\widehat{\mathrm{M}}^{(2)}_{ij}(t-\frac{r_*}{c}-v)+O(1/c^4),\\
r_*=r &+ \frac{2Gm}{c^2}\ln\left(\frac{r}{cb}\right)+O(1/c^3)
\end{aligned}
$$

Luminosity and energy loss

$$\mathcal{L}(t) = \frac{c^3}{32\pi G}\oint_{\mathrm{FZ}}(\partial_t h_{ij}^{\mathrm{TT}})^2 r^2 d\Omega$$

$$\begin{aligned}
\mathcal{L}(t) &= \frac{G}{5c^5}\sum_{n=0}^{\infty}\left(\frac{1}{c^2}\right)^n \hat{\mathcal{L}}_n(t) \\
&= \frac{G}{5c^5}\left\{\mathrm{M}_{ij}^{(3)}\mathrm{M}_{ij}^{(3)} + \frac{1}{c^2}\left[\frac{5}{189}\mathrm{M}_{ijk}^{(4)}\mathrm{M}_{ijk}^{(4)} + \frac{16}{9}\mathrm{S}_{ij}^{(3)}\mathrm{S}_{ij}^{(3)}\right]\right. \\
&+ \left.\frac{1}{c^4}\left[\frac{5}{9072}\mathrm{M}_{ijkm}^{(5)}\mathrm{M}_{ijkm}^{(5)} + \frac{5}{84}\mathrm{S}_{ijk}^{(4)}\mathrm{S}_{ijk}^{(4)}\right]\right\}
\end{aligned}$$

$$-<\frac{d\mathcal{E}(t_{\mathrm{ret}})}{dt}> \quad = \quad <\mathcal{L}(t)>$$

Hamiltonian for General Relativity

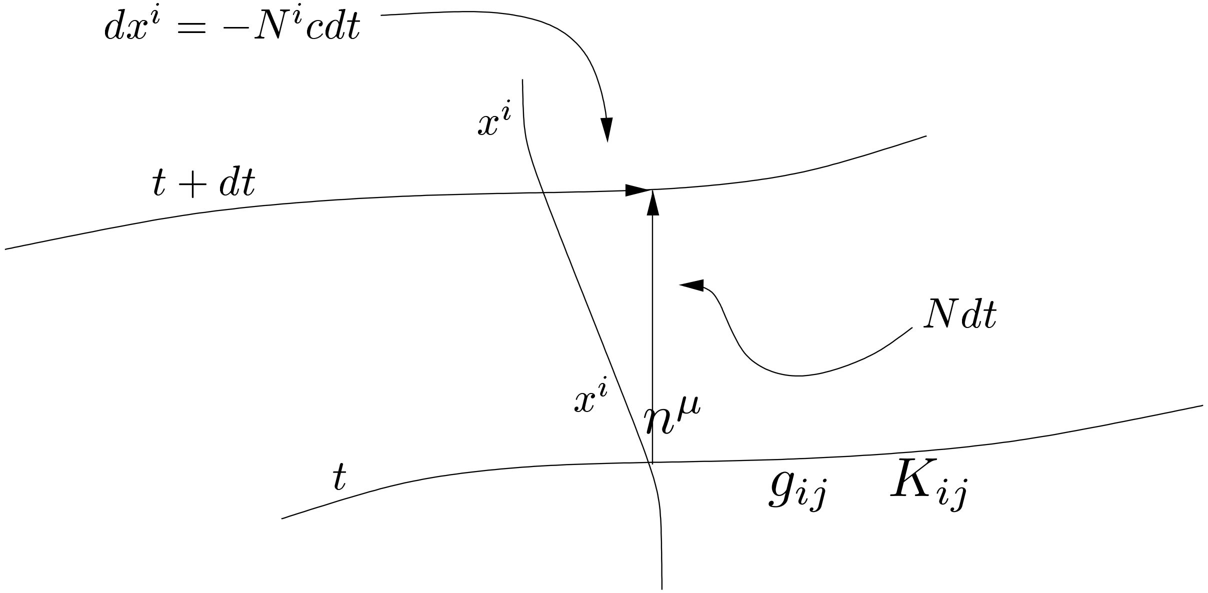
G. Schaefer
Post-Newtonian methods: Analytic results on the binary problem
arXiv:0910.2857

D. D. Holm
Hamiltonian formalism for general-relativistic adiabatic fluids
Physica **17D**, 1 (1985)

3+1 splitting of spacetime

$$n^\mu = (1, -N^i)/N$$

$$n_\mu = (-N, 0, 0, 0)$$



$$K_{ij} = -N\Gamma^0_{ij} = -Ng^{0\mu}(g_{i\mu,j} + g_{j\mu,i} - g_{ij,\mu})/2$$

$$\boxed{ds^2 = -(Ncdt)^2 + g_{ij}(dx^i + N^i cdt)(dx^j + N^j cdt)}$$

$$ds^2 = -(Ncdt)^2 + g_{ij}(dx^i + N^i cdt)(dx^j + N^j cdt)$$

$$H = \int d^3x(N\mathcal{H} - N^i\mathcal{H}_i) + \frac{c^4}{16\pi G}\oint_{i^0} d^2s_i(g_{ij,j} - g_{jj,i})$$

$$N|_{i^0} = 1 + \mathcal{O}(1/r), \quad N^i|_{i^0} = \mathcal{O}(1/r)$$

If the constraints $\mathcal{H} = 0$ and $\mathcal{H}_i = 0$ are fulfilled and adapted coordinate conditions happen, then

$$H = \frac{c^4}{16\pi G}\oint_{i^0} d^2s_i(g_{ij,j} - g_{jj,i}) \equiv H_{\mathrm{ADM}}$$

Binary Black-Hole Spacetimes

independent field variables

coordinate conditions:

$$g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}$$

$$\pi^{ii} = 0, \quad \pi^{ij} = -\gamma^{1/2}(K^{ij} - \gamma^{ij}K), \quad \pi_i^i = \pi^{ij}h_{ij}^{\mathrm{TT}},\, \gamma = \det(\gamma_{ij}),\, \gamma_{ij} = g_{ij}$$

unique decomposition: $\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\mathrm{TT}}$

$$\tilde{\pi}^{ij} = \partial_i \pi^j + \partial_j \pi^i - \frac{2}{3}\delta_{ij}\partial_k \pi^k$$

$\pi^{ij}_{\mathrm{TT}} c^3/16\pi G$: canonical conjugate to h_{ij}^{TT}

Hamilton and momentum constraints

$$\gamma^{1/2}\mathrm{R} = \frac{1}{\gamma^{1/2}}\left(\pi^i_j\pi^j_i - \frac{1}{2}\pi^i_i\pi^j_j\right) + \frac{16\pi G}{c^3}\sum_a \left(m_a^2c^2 + \gamma^{ij}p_{ai}p_{aj}\right)^{1/2}\delta_a$$

$G^{00} = \frac{8\pi G}{c^4}T^{00}$

$$-2\partial_j\pi^j_i + \pi^{kl}\partial_i\gamma_{kl} = \frac{16\pi G}{c^3}\sum_a p_{ai}\delta_a$$

$G^0_i = \frac{8\pi G}{c^4}T^0_i$

isolated BH

$$
\begin{aligned}
ds^2 &= -\left(\frac{1-\frac{Gm}{2rc^2}}{1+\frac{Gm}{2rc^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2rc^2}\right)^4 \delta_{ij}dx^i dx^j \\
&= -\left(\frac{1-\frac{Gm}{2Rc^2}}{1+\frac{Gm}{2Rc^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2Rc^2}\right)^4 \delta_{ij}dX^i dX^j
\end{aligned}
$$

symmetry transformation (inversion): $Rr = \left(\frac{Gm}{2c^2}\right)^2$

$R^2 = X^i X^i, \quad r^2 = x^i x^i$

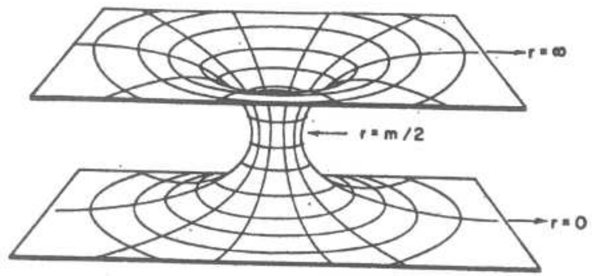


FIG. 1. A two-dimensional analog of the Schwarzschild-Kruskal manifold is shown isometrically imbedded in flat three-space. The figure shows the curvature and topology of the metric

$$ds^2 = (1+m/2r)^4(dr^2+r^2d\theta^2).$$

The sheets at the top and bottom of the funnel continue to infinity and represent the asymptotically flat regions of the manifold ($r\to 0$, $r\to\infty$).

Brill/Lindquist, JMP 1963

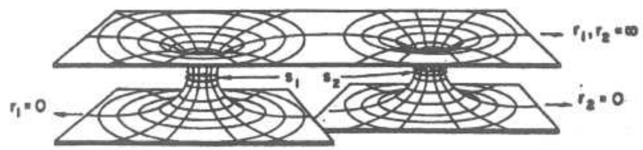
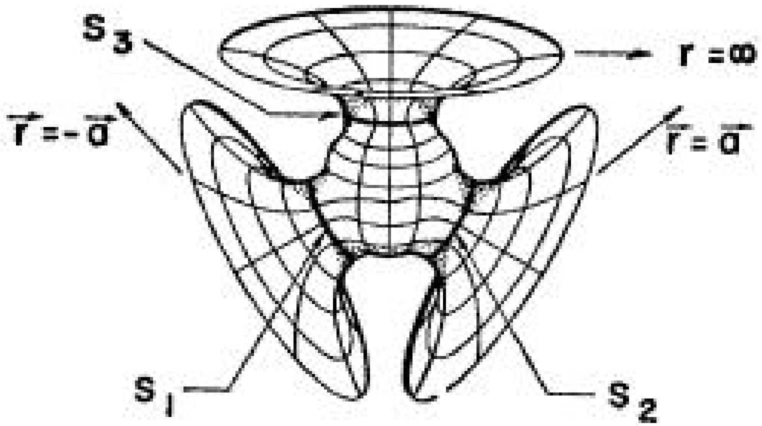
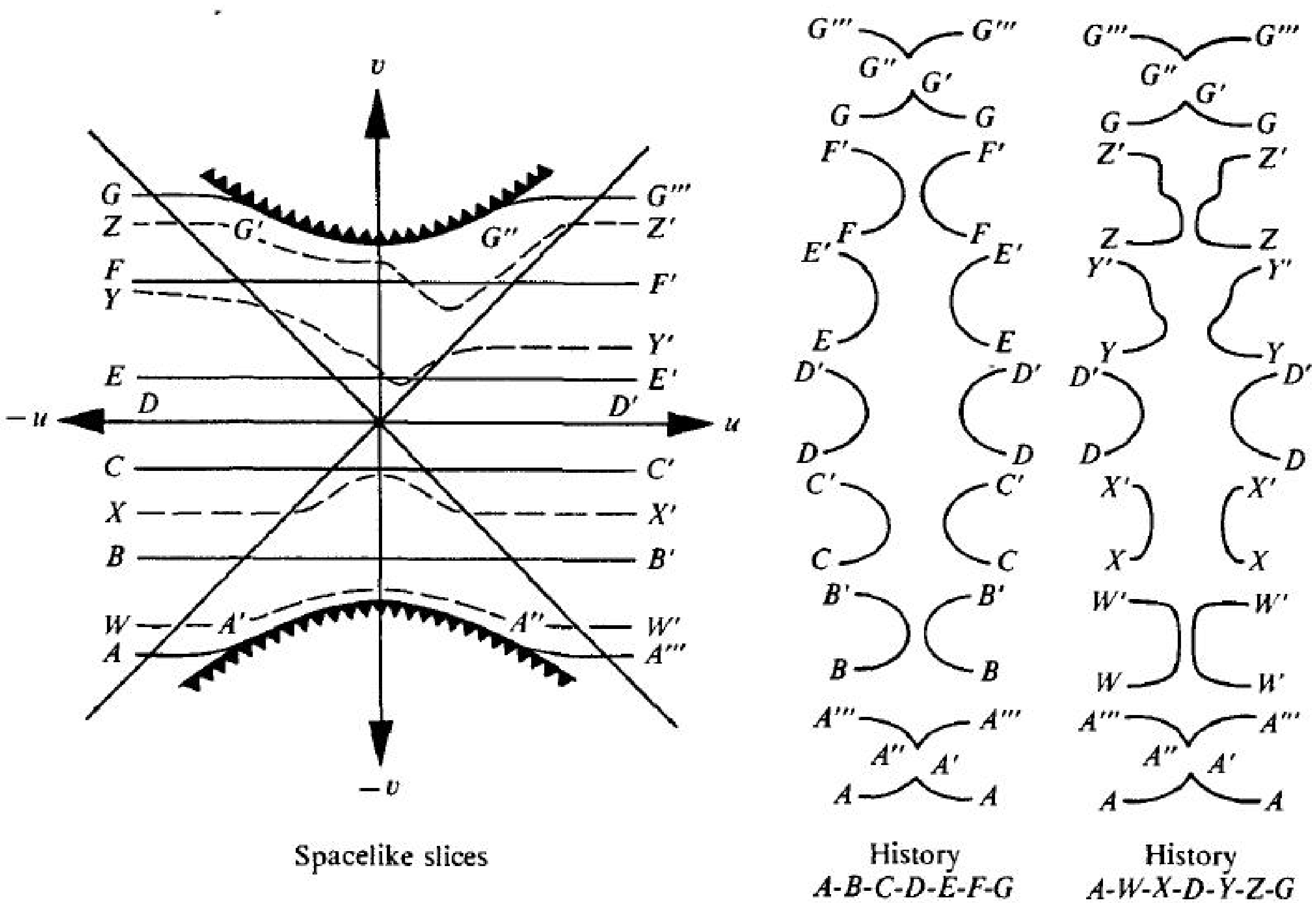


FIG. 2. A two-dimensional analog of the hypersurface of time symmetry of a manifold containing two "throats" is shown isometrically imbedded in flat three-space. The figure illustrates the curvature and topology for a system of two "particles" of equal mass m, and separation large compared to m, described by the metric

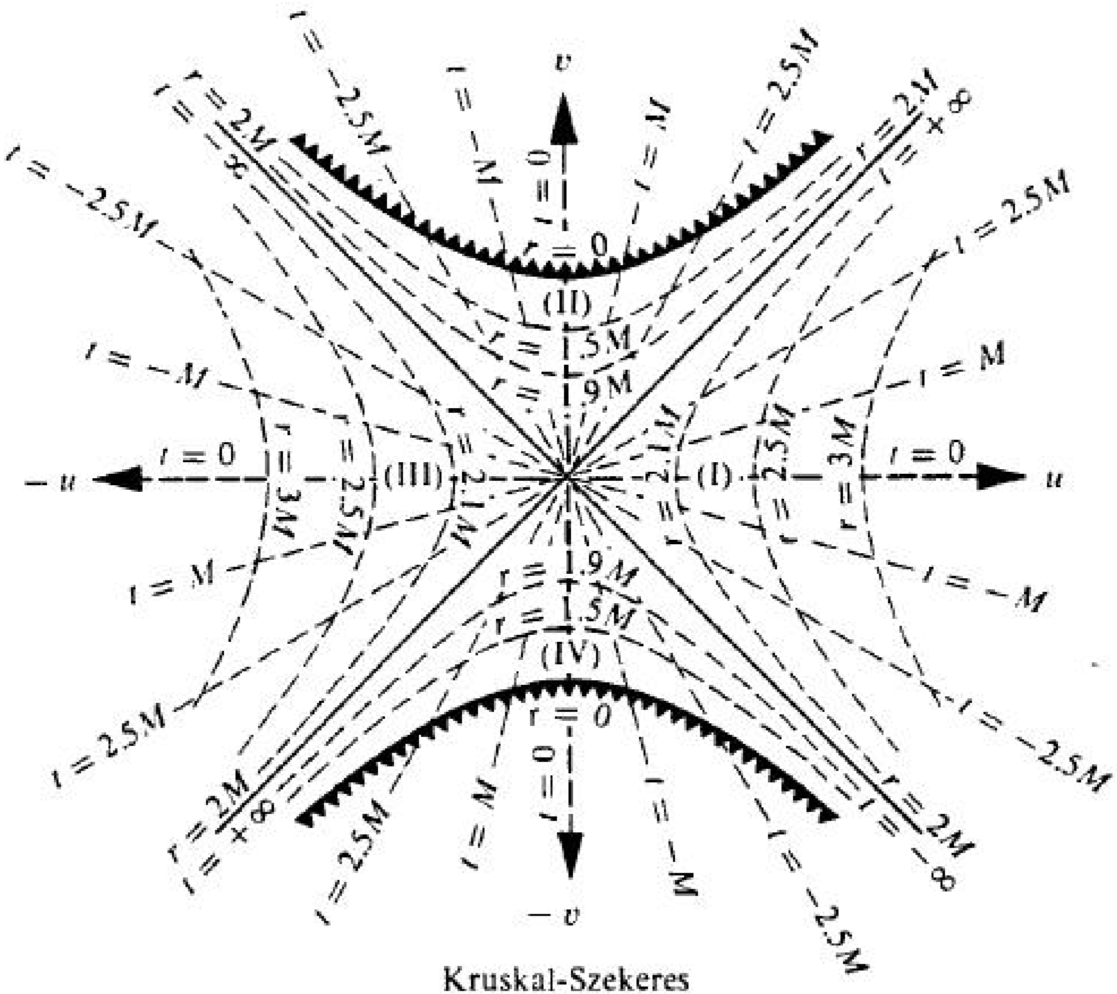
$$ds^2 = (1+m/2r_1+m/2r_2)^4 ds_F^2.$$



Brill-Lindquist: JMP 1963



MTW: Gravitation (1973)


Kruskal-Szekeres

MTW: Gravitation

Brill-Lindquist BHs

maximally sliced

$$ds^2 = -\left(\frac{1 - \frac{\beta_1 G}{2r_1c^2} - \frac{\beta_2 G}{2r_2c^2}}{1 + \frac{\alpha_1 G}{2r_1c^2} + \frac{\alpha_2 G}{2r_2c^2}}\right)^2 c^2dt^2 + \left(1 + \frac{\alpha_1 G}{2r_1c^2} + \frac{\alpha_2 G}{2r_2c^2}\right)^4 d\mathbf{x}^2$$

total energy: $$E_{ADM} = -\frac{c^4}{2\pi G}\oint_{i_0} ds_i \partial_i \Psi = (\alpha_1 + \alpha_2)c^2$$

$\Psi = 1 + \frac{\alpha_1 G}{2r_1c^2} + \frac{\alpha_2 G}{2r_2c^2}$

inversion map of the three-metric at the throat of black hole 1

$\mathbf{r}_1' = \mathbf{r}_1 \alpha_1^2 G^2 / 4c^4 r_1^2$

$\mathbf{r}_1' = \mathbf{x}' - \mathbf{x}_1, \quad \mathbf{r}_1 = \mathbf{x} - \mathbf{x}_1, \quad r_1 = |\mathbf{x} - \mathbf{x}_1|$

$$dl^2 = \Psi^4 d\mathbf{x}^2 = \left(1 + \frac{\alpha_1 G}{2r_1 c^2} + \frac{\alpha_2 G}{2r_2 c^2}\right)^4 d\mathbf{x}^2$$

$$dl^2 = \Psi'^4 d\mathbf{x}'^2 = \left(1 + \frac{\alpha_1 G}{2r'_1 c^2} + \frac{\alpha_1 \alpha_2 G^2}{4r_2 r'_1 c^4}\right)^4 d\mathbf{x}'^2$$

$$\mathbf{r}_2 = \frac{\alpha_1^2 G^2}{4c^4} \frac{\mathbf{r}'_1}{r'^2_1} + \mathbf{r}_{12}, \qquad \mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$$

$$\boxed{m_1 = -\frac{c^2}{2\pi G} \oint_{i_0} ds'_i \partial'_i \Psi' = \alpha_1 + \frac{\alpha_1 \alpha_2 G}{2r_{12} c^2}}$$

$$\Psi' = 1 + \frac{\alpha_1 G}{2r'_1 c^2} + \frac{\alpha_1 \alpha_2 G^2}{4r_2 r'_1 c^4}$$

$$-\left(1+\frac{1}{8}\;\phi\right)\Delta\phi=\frac{16\pi G}{c^2}\sum_a m_a\delta_a \qquad (h_{ij}^{\mathrm{TT}}=0=p_{ai})$$

$$\phi=\frac{4G}{c^2}\left(\frac{\alpha_1}{r_1}+\frac{\alpha_2}{r_2}\right)$$

$$\alpha_a=m_a-\frac{m_a+m_b}{2}+\frac{c^2r_{ab}}{G}\left(\sqrt{1+\frac{m_a+m_b}{c^2r_{ab}/G}+\left(\frac{m_a-m_b}{2c^2r_{ab}/G}\right)^2}-1\right)$$

$$\boxed{\mathrm{H}_{\mathrm{BL}}=(\alpha_1+\alpha_2)\;c^2=(m_1+m_2)\;c^2-G\;\frac{\alpha_1\alpha_2}{r_{12}}}$$

Metric in d-dimensional conformally flat space:

$$g_{ij} = \left(1 + \frac{1}{4}\frac{d-2}{d-1}\phi\right)^{\frac{4}{d-2}} \delta_{ij}$$

$$\phi = \frac{4G}{c^2} \frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d-2}{2}}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}}\right)$$

$$\Psi = 1 + \frac{1}{4}\frac{d-2}{d-1}\phi$$

$$-\Delta^{-1}\delta = \frac{\Gamma((d-2)/2)}{4\pi^{d/2}} r^{2-d}$$

$$\Psi = 1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}} \right)$$

$$\left(1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}} \right)\right) \alpha_1\delta_1 = m_1\delta_1$$

$1 < d < 2$

$$\left(1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \frac{\alpha_2}{r_{12}^{d-2}} \right) \alpha_1\delta_1 = m_1\delta_1$$

Higher Order Post-Newtonian Hamiltonians

$$\Box^{-1}_{\rm sym} = \left(1 + \frac{1}{c^2}\Delta^{-1}\partial_t^2 + ...\right)\Delta^{-1}\delta(t-t')$$

$$G_{\rm ret} = \left(1 - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\partial_t + \frac{1}{2c^2}|\mathbf{r}-\mathbf{r}'|^2\partial_t^2 + ...\right)\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\delta(t-t')$$

binary black holes to 4PN order

$$\begin{aligned} H(t) &= m_1c^2 + m_2c^2 + H_N + \frac{1}{c^2}H_{[1PN]} \\ &+ \frac{1}{c^4}H_{[2PN]} + \frac{1}{c^6}H_{[3PN]} + \frac{1}{c^8}H_{[4PN]} + \dots \\ &+ \frac{1}{c^5}H_{[2.5PN]}(t) + \frac{1}{c^7}H_{[3.5PN]}(t) + \dots \end{aligned}$$

$\hat{H} = (H - Mc^2)/\mu, \qquad \mu = m_1m_2/M, \qquad M = m_1 + m_2$

$\nu = \mu/M, \qquad 0 \le \nu \le 1/4$

test particles: $\nu = 0$, equal masses: $\nu = 1/4$

CMF: $\mathbf{p}_1 + \mathbf{p}_2 = 0, \qquad \mathbf{p} = \mathbf{p}_1/\mu, \qquad r = r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|,$

$p_r = (\mathbf{n}\cdot\mathbf{p}), \qquad \mathbf{q} = (\mathbf{x}_1 - \mathbf{x}_2)/GM, \qquad \mathbf{n} = \mathbf{n}_{12} = \mathbf{q}/|\mathbf{q}|$

$$\hat{H}_N = \frac{p^2}{2} - \frac{1}{q}$$

$$\hat{H}_{[1PN]} = \frac{1}{8}(-1+3\nu)p^4 - \frac{1}{2}[(3+\nu)p^2 + \nu p_r^2]\frac{1}{q} + \frac{1}{2q^2}$$

$$\begin{aligned}\hat{H}_{[2PN]} &= \frac{1}{16}(1-5\nu+5\nu^2)p^6 \\ &+ \frac{1}{8}[(5-20\nu-3\nu^2)p^4 - 2\nu^2 p_r^2 p^2 - 3\nu^2 p_r^4]\frac{1}{q} \\ &+ \frac{1}{2}[(5+8\nu)p^2 + 3\nu p_r^2]\frac{1}{q^2} - \frac{1}{4}(1+3\nu)\frac{1}{q^3}\end{aligned}$$

2.5PN dissipative binary dynamics

$$\hat{H}_{[2.5PN]}(\hat{t}) = \frac{2}{5}\frac{d^3 Q_{ij}(\hat{t})}{d\hat{t}^3}\left(p_i p_j - \frac{n^i n^j}{q}\right)$$

$$Q_{ij}(\hat{t}) = \nu(q'^i q'^j - \frac{1}{3}q'^2\delta_{ij})$$

$\hat{t} = t/GM$

$$
\begin{aligned}
\hat{H}_{[3PN]} &= \frac{1}{128}(-5+35\nu-70\nu^2+35\nu^3)p^8 \\
&+ \frac{1}{16}[(-7+42\nu-53\nu^2-5\nu^3)p^6+(2-3\nu)\nu^2p_r^2p^4 \\
&+ 3(1-\nu)\nu^2p_r^4p^2-5\nu^3p_r^6]\frac{1}{q} \\
&+ [\frac{1}{16}(-27+136\nu+109\nu^2)p^4+\frac{1}{16}(17+30\nu)\nu p_r^2p^2 \\
&+ \frac{1}{12}(5+43\nu)\nu p_r^4]\frac{1}{q^2} \\
&+ [\left(-\frac{25}{8}+\left(\frac{1}{64}\pi^2-\frac{335}{48}\right)\nu-\frac{23}{8}\nu^2\right)p^2 \\
&+ \left(-\frac{85}{16}-\frac{3}{64}\pi^2-\frac{7}{4}\nu\right)\nu p_r^2]\frac{1}{q^3}+[\frac{1}{8}+\left(\frac{109}{12}-\frac{21}{32}\pi^2\right)\nu]\frac{1}{q^4}
\end{aligned}
$$

$$
\begin{aligned}
\hat{H}_{[4PN]} \quad &= \quad \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right) p^{10} \\
&+ \quad \Bigg\{ \frac{45}{128}p^8 - \frac{45}{16}p^8\nu + \left(\frac{423}{64}p^8 - \frac{3}{32}p_r^2p^6 - \frac{9}{64}p_r^4p^4\right)\nu^2 \\
&+ \quad \left(-\frac{1013}{256}p^8 + \frac{23}{64}p_r^2p^6 + \frac{69}{128}p_r^4p^4 - \frac{5}{64}p_r^6 + \frac{35}{256}p_r^8\right)\nu^3 \\
&+ \quad \left(-\frac{35}{128}p^8 - \frac{5}{32}p_r^2p^6 - \frac{9}{64}p_r^4p^4 - \frac{5}{32}p_r^6 - \frac{35}{128}p_r^8\right)\nu^4 \Bigg\} \frac{1}{q} \\
&+ \quad \Bigg\{ \frac{13}{8}p^6 + \left(-\frac{791}{64}p^6 + \frac{49}{16}p_r^2p^4 - \frac{889}{192}p_r^4 + \frac{369}{160}p_r^6\right)\nu \\
&+ \quad \left(\frac{4857}{256}p^6 - \frac{545}{64}p_r^2p^4 + \frac{9475}{768}p_r^4 - \frac{1151}{128}p_r^6\right)\nu^2 \\
&+ \quad \left(\frac{2335}{256}p^6 + \frac{1135}{256}p_r^2p^4 - \frac{1649}{768}p_r^4 + \frac{10353}{1280}p_r^6\right)\nu^3 \Bigg\} \frac{1}{q^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \left[\frac{105}{32}p^4 + C_{41}\,\nu + C_{42}\,\nu^2 + \left(-\frac{553}{128}p^4 - \frac{225}{64}p_r^2 - \frac{381}{128}p_r^4\right)\nu^3\right]\frac{1}{q^3} \\
&+ \quad \left\{\frac{105}{32} + C_{21}\,\nu + C_{22}\,\nu^2\right\}\frac{1}{q^4} \\
&+ \quad \left\{-\frac{1}{16} + c_{01}\,\nu + c_{02}\,\nu^2\right\}\frac{1}{q^5} \\
&- \quad \frac{1}{5}\hat{I}_{ij}^{(3)}(\hat{t}) \int_{-\infty}^{+\infty} dw \ln\left(\frac{|w|c}{2q}\right) \hat{I}_{ij}^{(4)}(\hat{t}-w)\,\nu
\end{aligned}
$$

$$
\begin{aligned}
C_{42} &= \left(-\frac{1189789}{28800} + \frac{18491}{16384}\pi^2\right) p^4 + \left(-\frac{127}{3} - \frac{4035}{2048}\pi^2\right) p_r^2 p^2 \\
&+ \left(\frac{57563}{1920} - \frac{38655}{16384}\pi^2\right) p_r^4
\end{aligned}
$$

$$
C_{22} = \left(\frac{672811}{19200} - \frac{158177}{49152}\pi^2\right) p^2 + \left(-\frac{21827}{3840} + \frac{110099}{49152}\pi^2\right) p_r^2
$$

$$
c_{02} = -\frac{1256}{45} + \frac{7403}{3072}\pi^2
$$

$$C_{41} = \left(-\frac{589189}{19200} + \frac{2749}{8192}\pi^2\right) p^4 + \left(\frac{63347}{1600} - \frac{1059}{1024}\pi^2\right) p_r^2 p^2$$

$$+ \left(-\frac{23533}{1280} + \frac{375}{8192}\pi^2\right) p_r^4$$

$$C_{21} = \left(\frac{185761}{19200} - \frac{21837}{8192}\pi^2\right) p^2 + \left(\frac{3401779}{57600} - \frac{28691}{24576}\pi^2\right) p_r^2$$

$$c_{01} = -\frac{169199}{2400} + \frac{6237}{1024}\pi^2$$

4PN

Jaranowski/GS (2013,2015), Damour/Jaranowski/GS (2014)

$$
\begin{aligned}
H_{4\mathrm{PN}}^{\mathrm{near-zone}\,(\mathrm{s})}[\mathbf{x}_a, \mathbf{p}_a] &= H_{4\mathrm{PN}}^{\mathrm{loc}\,0}[\mathbf{x}_a, \mathbf{p}_a] \\
&+ \frac{2}{5}\,\frac{G^2 M}{c^8}\big(I_{ij}^{(3)}\big)^2 \left(\ln\,\frac{r_{12}}{s} + C\right) \\
&+ \frac{d}{dt}\,G[\mathbf{x}_a, \mathbf{p}_a]
\end{aligned}
$$

$$
\begin{aligned}
H_{4\mathrm{PN}}^{\mathrm{tailsym}\,(\mathrm{s})}(t) &= -\frac{1}{5}\,\frac{G^2 M}{c^8}\,I_{ij}^{(3)}(t) \\
&\times \int_{-\infty}^{+\infty} dv \ln\left(\frac{|v|c}{2s}\right) I_{ij}^{(4)}(t-v)
\end{aligned}
$$

Matching to results by Bini/Damour for perturbed Schwarzschild metric from particle in circular motion yields $C = -\frac{1681}{1536}$.

$$H_{4\mathrm{PN}}^{\mathrm{tailsym}}(t) = -\frac{1}{5}\,\frac{G^2 M}{c^8}\, I_{ij}^{(3)}(t)\,\mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|}\, I_{ij}^{(3)}(t-v)$$

Classical gravitational “Lamb Shift” (orbital average):

$$\Delta E = -\frac{G}{5c^5}\,\frac{GM}{c^3 P}\int_0^P dt\left[I_{ij}^{(3)}(t)\,\mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|}\, I_{ij}^{(3)}(t-v)\right]$$

$$< \frac{dE}{dt} > = -\frac{G}{5c^5}\,\frac{1}{P}\int_0^P dt\left[I_{ij}^{(3)}(t) I_{ij}^{(3)}(t)\right]$$ (Einstein’s quad. formula)

Orbital Motion (ISCO)

ISCO

$$H = H(\mathbf{p}, \mathbf{r}), \qquad p^2 = p_r^2 + j^2/r^2, \qquad p_r = (\mathbf{p}\cdot\mathbf{r})/r$$

circular orbits: $p_r = 0, \qquad p^2 = j^2/r^2, \qquad H = H(j, r)$

circular motion: $\frac{\partial}{\partial r} H(j, r) = 0 \rightarrow H(j)$

orbital frequency: $\omega = \frac{dH(j)}{dj} \rightarrow H(\omega)$

ISCO: $\boxed{\frac{dH(\omega)}{d\omega} = 0}$ or, alternatively $\frac{\partial^2}{\partial r^2} H(j, r) = 0$

$$\text{SBH:} \quad E(x) = \frac{1-2x}{(1-3x)^{1/2}} - 1$$

$$= -\frac{1}{2}x + \frac{3}{8}x^2 + \frac{27}{16}x^3 + \frac{675}{128}x^4 + \frac{3969}{256}x^5 + \ldots$$

$$E(x) \equiv \frac{H(x) - mc^2}{mc^2}, \qquad x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$$

circular orbits:

$$c^2 E_{4PN} \equiv \hat{H}_N + c^{-2}\hat{H}_{[1PN]} + c^{-4}\hat{H}_{[2PN]} + c^{-6}\hat{H}_{[3PN]} + c^{-8}\hat{H}_{[4PN]}$$

$$\begin{aligned} E_{4PN}(x) &= -\frac{x}{2} + \left(\frac{3}{8} + \frac{1}{24}\nu\right)x^2 + \left(\frac{27}{16} - \frac{19}{16}\nu + \frac{1}{48}\nu^2\right)x^3 \\ &+ \left(\frac{675}{128} + \left(-\frac{34445}{1152} + \frac{205}{192}\pi^2\right)\nu + \frac{155}{192}\nu^2 + \frac{35}{10368}\nu^3\right)x^4 \end{aligned}$$

$$-\tfrac{1}{2}\left(-\tfrac{3960}{128} + [c_1 + \tfrac{448}{15}\ln x]\nu + \left(-\tfrac{498449}{3456} + \tfrac{3157}{576}\pi^2\right)\nu^2 + \tfrac{301}{1728}\nu^3 + \tfrac{77}{31104}\nu^4\right)x^5$$

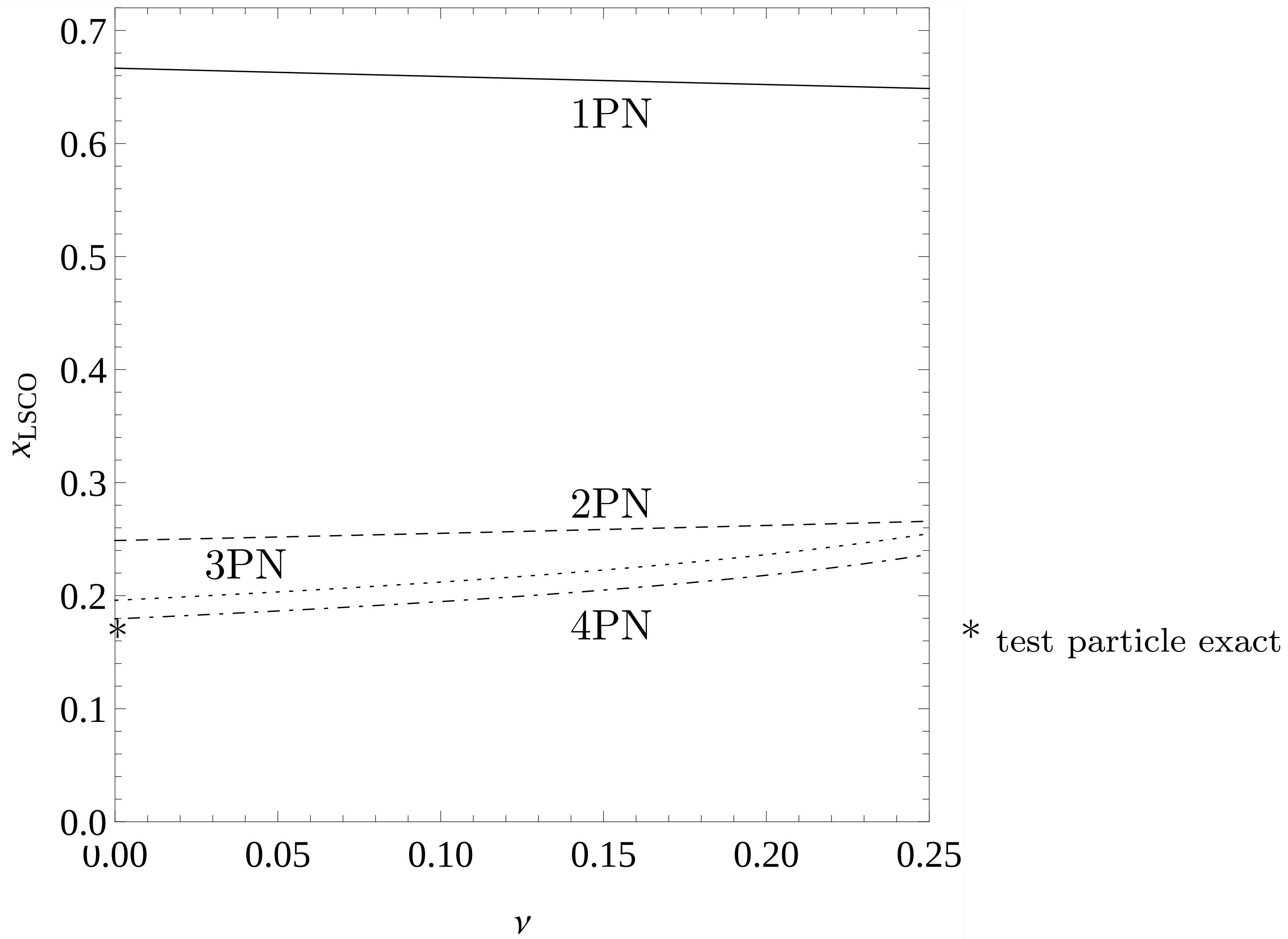
Damour ('10)[ln$x$], Blanchet/Detweiler/Le Tiec/Whiting ('10)[ln$x$]

Jaranowski/GS ('12)[ln$x$, ν^3, ν^4], ('13)[ν^2], Foffa/Sturani ('13) [ln$x$, ν^3, ν^4]

$$c_1 = -\tfrac{123671}{5760} + \tfrac{9037}{1536}\pi^2 + \tfrac{1792}{15}\ln 2 + \tfrac{896}{15}\gamma_E = 153.88...$$

Bini/Damour ('13), Le Tiec/Blanchet/Whiting ('12) [numerical value]

Jaranowski/GS (2013)



Spin-Gravity Interaction

leading order spin orbit

$$H_{\rm SO}^{\rm LO} = \frac{G}{c^2}\sum_a \sum_{b\neq a} \frac{1}{r_{ab}^2} (\mathbf{S}_a \times \mathbf{n}_{ab}) \cdot \left[\frac{3m_b}{2m_a}\mathbf{p}_a - 2\mathbf{p}_b \right]$$

leading order spin(1)-spin(2)

$$H_{\rm S_1S_2}^{\rm LO} = \frac{G}{c^2}\sum_a \sum_{b\neq a} \frac{1}{2r_{ab}^3} \left[3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b) \right]$$

leading order spin(1) spin(1)

$$H_{\rm S_1^2}^{\rm LO} = \frac{G}{c^2}\frac{m_2}{2m_1 r_{12}^3} C_{Q_1} \left[3(\mathbf{S}_1 \cdot \mathbf{n}_{12})(\mathbf{S}_1 \cdot \mathbf{n}_{12}) - (\mathbf{S}_1 \cdot \mathbf{S}_1) \right]$$

$$
\begin{aligned}
H_{\mathrm{con}} &= H_N + H_{1PN} + H_{2PN} + H_{3PN} + H_{4PN} \\
&+ H^{\mathrm{LO}}_{SO} + H^{\mathrm{LO}}_{S_1 S_2} + H^{\mathrm{LO}}_{S_1^2} + H^{\mathrm{LO}}_{S_2^2} \\
&+ H^{\mathrm{NLO}}_{SO} + H^{\mathrm{NLO}}_{S_1 S_2} + H^{\mathrm{NLO}}_{S_1^2} + H^{\mathrm{NLO}}_{S_2^2} \\
&+ H^{\mathrm{NNLO}}_{SO} + H^{\mathrm{NNLO}}_{S_1 S_2} + H^{\mathrm{NNLO}}_{S_1^2} + H^{\mathrm{NNLO}}_{S_2^2} \\
&+ \sum_{a,b,c\,=\,1,2} H^{\mathrm{LO}}_{p_a S_b S_c^2} \\
&+ H^{\mathrm{LO}}_{S_1^2 S_2^2} + H^{\mathrm{LO}}_{S_1 S_2^3} + H^{\mathrm{LO}}_{S_2 S_1^3} + H^{\mathrm{LO}}_{S_1^4} + H^{\mathrm{LO}}_{S_1^4} \\
H_{\mathrm{diss}}(t) &= H_{2.5PN}(t) + H_{3.5PN}(t) \\
&+ H^{\mathrm{DLO}}_{SO}(t) + H^{\mathrm{DLO}}_{S_1 S_2}(t)
\end{aligned}
$$

e.g., Steinhoff ('11), Hartung/Steinhoff/GS ('13), Levi/Steinhoff ('16)

Data Analysis

S. Finn: *Detection, measurement, and gravitational radiation*, PRD 46, 5236 (1992)

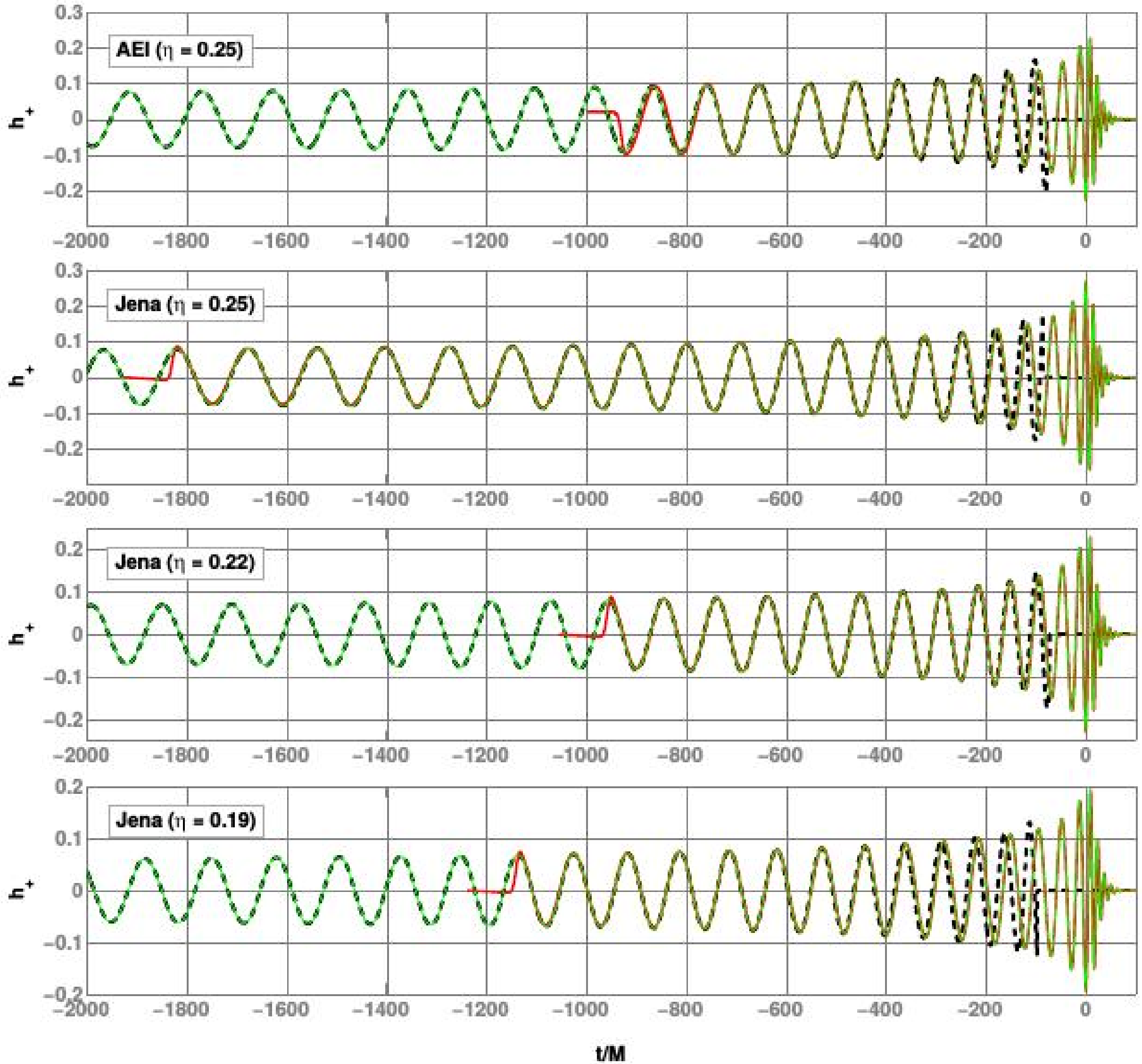
C. Cutler/E. Flanagan: *Gravitational waves from merging compact binaries: How accurately can one extract the binary's parameters from the inspiral waveform?*, PRD 49, 2658 (1994)

A. Królak/K. Kokkotas/G. Schäfer: *Estimation of the post-Newtonian parameters in the gravitational-wave emission of a coalescing binary*, PRD 52, 2089 (1995)

J. Creighton/W. Anderson: *Gravitational-Wave Physics and Astronomy*, WILEY-VCH Verlag, Weinheim 2011

3.5PN GW in comparison with IMRPhenom and NR

Ajith/Babak/Chen et al. 2008



AEI (η = 0.25)
Jena (η = 0.25)
Jena (η = 0.22)
Jena (η = 0.19)
h+
t/M

fundamental inner product for data analysis:

$$< h_1|h_2 > \quad = 2\int_0^\infty \frac{\tilde{h}_1^*(f)\tilde{h}_2(f) + \tilde{h}_2^*(f)\tilde{h}_1(f)}{S_n(f)} = 4\mathrm{Re}\int_0^\infty \frac{\tilde{h}_1^*(f)\tilde{h}_2(f)}{S_n(f)}$$

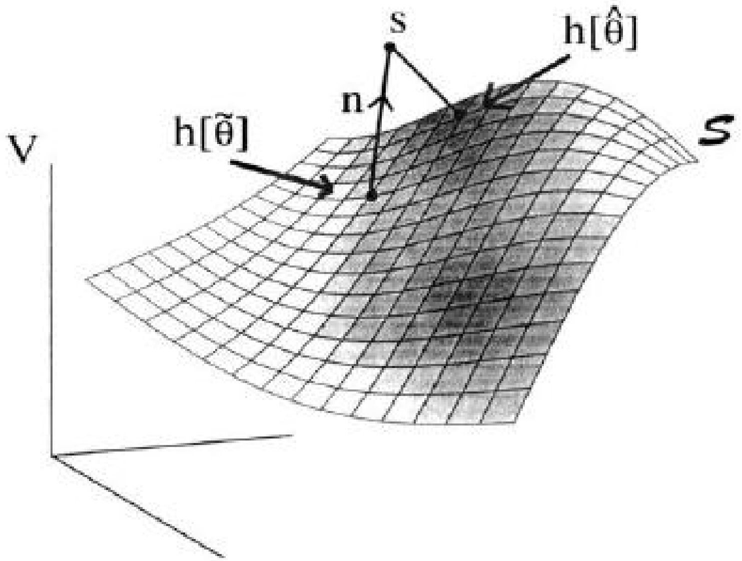


FIG. 1. Gravitational waveforms from coalescing compact binaries are completely specified by a finite number of parameters $\theta = (\theta^1, \ldots, \theta^k)$, and so form a surface $\mathcal{S}$ in the vector space V of all possible measured detector outputs $s = s(t)$. The statistical properties of the detector noise endow V with the structure of an infinite-dimensional Euclidean space. This figure illustrates the relationships between the true gravitational wave signal $h(\tilde{\theta})$, the measured signal s, and the "best-fit" signal $h(\hat{\theta})$. Given a measured detector output $s = h(\tilde{\theta}) + n$, where $n = n(t)$ is the detector noise, the most likely values $\hat{\theta}$ of the binaries parameters are just those that correspond to the point $h(\hat{\theta})$ on the surface $\mathcal{S}$ which is closest [in the Euclidean distance $(s - h \,|\, s - h)$] to y.

signal-to-noise ratio (of a signal): $(\hat{h} = h(\hat{\theta}))$

$$\frac{S}{N}[\hat{h}] = \frac{< \hat{h}|\hat{h} >}{\text{rms} < \hat{h}|n >} =< \hat{h}|\hat{h} >^{1/2} \equiv \rho$$

Measurement Sensitivity

Consider observed $s(t)$ which contains a signal $h(t;\tilde{\theta})$ for unknown $\tilde{\theta}$.

Interested in the distribution of $\Delta\theta = \tilde{\theta} - \hat{\theta}$.

$$< \frac{\partial h}{\partial \theta_i}(\hat{\theta}) | \frac{\partial h}{\partial \theta_j}(\hat{\theta}) > \quad \equiv \Gamma_{ij}$$ **Fisher information matrix**

root-mean-square error:

$\sqrt{\overline{(\Delta\theta_i)^2}} = \sqrt{\Sigma_{ii}}, \qquad \Sigma_{ij} = (\Gamma^{-1})_{ij},$ **variance-covariance matrix**

correlation coefficient: $c_{ij} = \Sigma_{ij} / \sqrt{\Sigma_{ii}\Sigma_{jj}}$

Application: Inspiral Waveform

$G = c = 1$

Circular motion, Newtonian approximation

$M = M_1 + M_2$, $\mu = M_1 M_2/M$, $\Omega = M^{1/2}/r^{3/2}$ (orbit), $f = \Omega/\pi$

$$\frac{dr}{dt} = -\frac{r}{E}\frac{dE}{dt} = -\frac{64}{5}\frac{\mu M^2}{r^3}$$

$$r = \left(\frac{256}{5}\mu M^2\right)^{1/4} (t_c - t)^{1/4}$$

$$h(t) = \frac{(384/5)^{1/2}\pi^{2/3} Q(\vartheta, \varphi, \psi, \iota)\mu M}{Dr(t)} \cos\left(\int^t 2\pi f(t')dt'\right)$$

$$\frac{df}{dt} = \frac{96}{5}\pi^{8/3}\mathcal{M}^{5/3} f^{11/3}, \qquad \mathcal{M} = \mu^{3/5} M^{2/5} \quad \textbf{chirp mass}$$

$$\phi(t) = \int^t 2\pi f(t')dt' = -2\left[\frac{t_c - t}{5\mathcal{M}}\right]^{5/8} + \phi_c$$

redshifted masses: $\mathcal{M} = (1+z)\mathcal{M}_{\rm true}, \qquad \mu = (1+z)\mu_{\rm true}$

luminosity distance $D = D_L$

Fourier transform: $\tilde{h}(f) = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi i f t} h(t) dt$

stationary phase approximation:

$B(t) = A(t)\mathrm{cos}\phi(t)$

$d\mathrm{ln}A/dt << d\phi/dt$ and $d^2\phi/dt^2 << (d\phi/dt)^2$

$$\tilde{B}(f) \approx \frac{1}{2} A(t) \left(\frac{df}{dt}\right)^{-1/2} \exp[i(2\pi f t - \phi(t) - \pi/4)], \qquad f \geq 0$$

$d\phi(t)/dt = 2\pi f$

$t(f) = t_c - 5(8\pi f)^{-8/3}\mathcal{M}^{-5/3}, \qquad \phi(t(f)) = \phi(f) = \phi_c - 2(8\pi f \mathcal{M})^{-5/3}$

$$\tilde{h}(f) = \frac{Q}{D}\mathcal{M}^{5/6} f^{-7/6} \exp[i\Psi(f)], \qquad f \geq 0$$

$$\Psi(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3}{4}(8\pi\mathcal{M}f)^{-5/3}$$

shut off at $r = 6M$ (about ISCO)

$\tilde{h}(f) = 0$ for $f > 1/(6^{3/2}\pi M)$

$$\frac{|dr/dt|}{r d\phi/dt} = \frac{2}{3}\frac{d^2\phi/dt^2}{(d\phi/dt)^2} < \frac{1}{55}\left(\frac{4\mu}{M}\right)$$

signal-to-noise squared:

$$(S/N)^2(f) = 4 \int_0^f \frac{|\tilde{h}(f')|^2}{S_n(f')} df' = 4 \frac{Q^2}{D^2} \mathcal{M}^{5/3} \int_0^f \frac{(f')^{-7/3}}{S_n(f')} df'$$

$S_n(f) = \infty$ for $f < 10$ Hz

for $f > 10$ Hz:

$$S_n(f) = S_0 \left[\left(\frac{f_0}{f} \right)^4 + 2 \left(1 + \left(\frac{f}{f_0} \right)^2 \right) \right]$$

$S_0 = 3 \times 10^{-48}\ \mathrm{Hz}^{-1}$, $f_0 = 70$ Hz

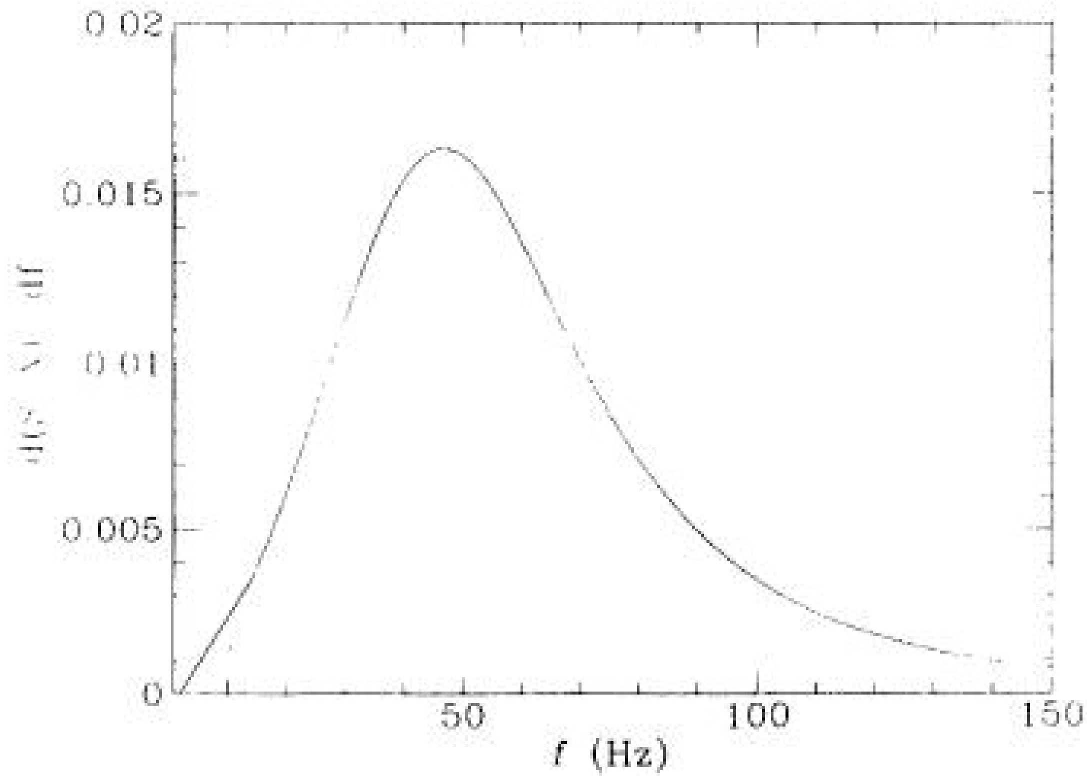


FIG. 2. This plot shows how the total signal-to-noise squared S^2/N^2 for a detected coalescing-binary waveform is distributed in frequency f, assuming the detector noise curve (2.1). Most of the signal-to-noise ratio comes not near 70 Hz where the detector sensitivity $S_n(f)^{-1}$ is highest, but rather at a somewhat lower frequency of ~ 50 Hz, because more cycles per unit frequency are received at lower frequencies.

Fisher information matrix: with $\mathcal{A} = (Q/D)\mathcal{M}^{5/6}$

$$\frac{\partial \tilde{h}}{\partial \ln \mathcal{A}} = \tilde{h}, \qquad \frac{\partial \tilde{h}}{\partial t_c} = 2\pi i f \tilde{h}$$

$$\frac{\partial \tilde{h}}{\partial \ln \mathcal{M}} = -\frac{5i}{4}(8\pi\mathcal{M}f)^{-5/3}\tilde{h}, \qquad \frac{\partial \tilde{h}}{\partial \phi_c} = -i\tilde{h}$$

NS-NS binaries:

$$\Delta(\ln \mathcal{A}) = 0.10\left(\frac{10}{S/N}\right), \qquad \Delta t_c = 0.40\left(\frac{10}{S/N}\right)\text{ms}$$

$$\Delta(\ln \mathcal{M}) = 1.2\times 10^{-5}\left(\frac{10}{S/N}\right)\left(\frac{\mathcal{M}}{M_\odot}\right)^{5/3}, \qquad \Delta\phi_c = 0.25\left(\frac{10}{S/N}\right)\text{rad}$$

BH-BH binaries with $S/N = 10$: $\Delta t_c = 0.60$ ms, $\Delta\phi_c = 0.32$ rad, $\Delta(\ln\mathcal{M}) = 1.3 \times 10^{-5}\left(\frac{\mathcal{M}}{M_\odot}\right)^{5/3}$

Post-Newtonian effects and parameter estimation

$$h(t) = \mathrm{Re}\left\{\sum_{x,m} h_m^x(t)\mathrm{e}^{im\sum_y \Phi^y(t)}\right\}$$

$$h_m^x(t) = \frac{\mu M}{Dr(t)} g_m^x(M_1/M_2) Q_m^x(\vartheta, \varphi, \psi, \iota)$$

$$\Phi(t) = \Phi^0(t) + \Phi^1(t) + \Phi^{1.5}(t) + ...$$

sufficient

$$h(t) = \mathrm{Re}\left\{h_2^0(t)\mathrm{e}^{i2\Phi(t)}\right\}$$

$h_2^0(t)$: Newtonian quadrupole amplitude

without spin but with tail

$$\Omega = \frac{M^{1/2}}{r^{3/2}}\left[1 + \left(-\frac{3}{2} + \frac{\mu}{2M}\right)\frac{M}{r}\right]$$

$$\frac{df}{dt} = \frac{96}{5}\pi^{8/3}\mathcal{M}^{5/3}f^{11/3}\left[1 - \left(\frac{743}{336} + \frac{11\mu}{4M}\right)(\pi M f)^{2/3} + 4\pi(\pi M f)\right]$$

$$t(f) = t_c - 5(8\pi f)^{-8/3}\mathcal{M}^{-5/3}\left[1 + \frac{4}{3}\left(\frac{743}{336} + \frac{11\mu}{4M}\right)x - \frac{32\pi}{5}x^{3/2}\right]$$

$$\phi(f) = \phi_c - 2(8\pi f\mathcal{M})^{-5/3}\left[1 + \frac{5}{3}\left(\frac{743}{336} + \frac{11\mu}{4M}\right)x - 10\pi x^{3/2}\right]$$

$x = (\pi M f)^{2/3}$

$$\tilde{h}(f) = \mathcal{A} f^{-7/6} \exp[i\Psi(f)], \qquad 0 < f < (6^{3/2}\pi M)^{-1}$$

$$\Psi(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3}{4}(8\pi\mathcal{M}f)^{-5/3}\left[1 + \frac{20}{9}\left(\frac{743}{336} + \frac{11\mu}{4M}\right)x - 16\pi x^{3/2}\right]$$

$$\frac{\partial\tilde{h}(f)}{\partial \ln\mathcal{M}} = -\frac{5i}{4}(8\pi\mathcal{M}f)^{-5/3}\tilde{h}(f)\left[1 + \frac{55\mu}{6M}x + 8\pi x^{3/2}\right]$$

$$\frac{\partial\tilde{h}(f)}{\partial \ln\mu} = \frac{3i}{4}(8\pi\mathcal{M}f)^{-5/3}\tilde{h}(f)\left[\left(-\frac{3715}{756} + \frac{55\mu}{6M}\right)x + 24\pi x^{3/2}\right]$$

TABLE I. The rms errors for signal parameters and the correlation coefficient $c_{\mathcal{M}\mu}$, calculated assuming spins are negligible. The results are for a single "advanced" detector, the shape of whose noise curve is given by Eq. (2.1). M_1 and M_2 are in units of solar masses, while Δt_c is in units of msec. The rms errors are normalized to a signal-to-noise ratio of $S/N = 10$; the errors scale as $(S/N)^{-1}$, while $c_{\mathcal{M}\mu}$ is independent of S/N.

M_1	M_2	$\Delta\phi_c$	Δt_c	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\mu/\mu$	$c_{\mathcal{M}\mu}$
2.0	1.0	1.31	0.721	0.0038%	0.39%	0.899
1.4	1.4	1.28	0.713	0.0040%	0.41%	0.906
10	1.4	1.63	1.01	0.020%	0.54%	0.927
15	5.0	2.02	1.44	0.113%	1.5%	0.954
10	10	1.98	1.43	0.16%	1.9%	0.958

with spin

$$\frac{df}{dt} = \frac{96}{5}\pi^{8/3}\mathcal{M}^{5/3}f^{11/3}\left[1 - \left(\frac{743}{336} + \frac{11\mu}{4M}\right)x + (4\pi - \beta)x^{3/2}\right]$$

$$\beta = M^{-2}\hat{\mathbf{L}} \cdot \left[\left(\frac{113}{12} + \frac{25}{4}\frac{M_2}{M_1}\right)\mathbf{S}_1 + \left(\frac{113}{12} + \frac{25}{4}\frac{M_1}{M_2}\right)\mathbf{S}_2\right]$$

β approximately constant

$\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2 = \text{const}$

$\Psi(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3}{4}(8\pi\mathcal{M}f)^{-5/3}\left[1 + \frac{20}{9}\left(\frac{743}{336} + \frac{11\mu}{4M}\right)x + (4\beta - 16\pi)x^{3/2}\right]$

$\mathcal{A} = Q(\vartheta, \varphi, \psi, \iota)D^{-1}\mathcal{M}^{5/6}$

$$\frac{\partial \tilde{h}(f)}{\partial \ln \mathcal{M}} = -\frac{5i}{4}(8\pi \mathcal{M} f)^{-5/3}\tilde{h}(f)\left[1 + \frac{55\mu}{6M}x + (8\pi - 2\beta)x^{3/2}\right]$$

$$\frac{\partial \tilde{h}(f)}{\partial \ln \mu} = \frac{3i}{4}(8\pi \mathcal{M} f)^{-5/3}\tilde{h}(f)\left[\left(-\frac{3715}{756} + \frac{55\mu}{6M}\right)x + (24\pi - 6\beta)x^{3/2}\right]$$

$$\frac{\partial \tilde{h}(f)}{\partial \beta} = 3i(8\pi \mathcal{M} f)^{-5/3}\tilde{h}(f)x^{3/2}$$

TABLE II. The rms errors for signal parameters and the correlation coefficients $c_{\mathcal{M}\mu}$, $c_{\mathcal{M}\beta}$, and $c_{\mu\beta}$, calculated using spin-dependent waveforms. The results are for a single "advanced" detector, the shape of whose noise curve is given by Eq. (2.1). For the rows marked with a † (and *only* for those rows), the variance-covariance matrix has been "corrected" to approximately account for the fact that the spin parameter β must satisfy $|\beta| < \beta_{\rm max} \approx 8.5$. The rms errors are normalized to a signal-to-noise ratio of $S/N = 10$. Except for rows marked with a †, errors scale as $(S/N)^{-1}$, while the correlation coefficients are independent of S/N. Except for rows marked with a †, if β had been chosen nonzero with M_1 and M_2 unchanged, then $\Delta\mathcal{M}/\mathcal{M}$, $\Delta\mu/\mathcal{M}$, and $c_{\mathcal{M}\mu}$ would have been unchanged (but $\Delta\beta$, $c_{\mathcal{M}\beta}$, and $c_{\mu\beta}$ would have been altered). As in Table I, M_1 and M_2 are in units of $M_\odot$, while Δt_c is in msec. The results for the LIGO-VIRGO network of detectors, for a signal with combined signal-to-noise ratio from all the detectors of 10, will be approximately the same as those shown here; see text.

M_1	M_2	β	$\Delta\phi_c$	Δt_c	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\mu/\mu$	$\Delta\beta$	$c_{\mathcal{M}\mu}$	$c_{\mathcal{M}\beta}$	$c_{\mu\beta}$
2.0	1.0	0	4.13	1.14	0.034%	8.44%	1.04	-0.988	0.993	-0.9989
1.4	1.4	0	4.09	1.13	0.034%	9.65%	1.24	-0.988	0.993	-0.9991
10	1.4	0	6.24	2.03	0.19%	15.2%	1.99	-0.990	0.994	-0.9994
5	1.4	0	4.89	1.44	0.10%	13.4%	1.73	-0.989	0.994	-0.9992
15	5	0	9.26	3.53	1.06%	76.4%	11.4	-0.992	0.994	-0.99980
15	5	0 †	5.77	2.40	0.64%	45.8%	6.81	-0.978	0.984	-0.9995
10	10	0	9.26	3.53	1.42%	125%	19.5	-0.992	0.994	-0.99988
10	10	0 †	4.13	1.92	0.59%	49.9%	7.79	-0.953	0.964	-0.9992

Appendix: On non-local-in-time Hamiltonian

Damour/Jaranowski/GS, PRD **89**, 064058 (2014)

Damour/Jaranowski/GS, PRD **93**, 084014 (2016)

Bernard/Blanchet/Bohé/Faye/Marsat, PRD **93**, 084037 (2016)

Bernard/Blanchet/Bohé/Faye/Marsat, arXiv:1610.07934

$$S^{\rm tail} = \frac{G^2 M}{5c^8} \int dt' I^{(3)}_{ij}(t') \mathrm{Pf}_{2r(t')/c} \int dt'' \frac{1}{|t'-t''|} I^{(3)}_{ij}(t'')$$

$S^{\rm tail} = -\int H^{\rm tail}(t) dt$

$$H^{\rm tail}(t) = -\frac{G^2 M}{5c^8} I^{(3)}_{ij}(t) \mathrm{Pf}_{2r(t)/c} \int_{-\infty}^{\infty} \frac{dv}{|v|} I^{(3)}_{ij}(t+v)$$

How to functionally differentiate $S^{\rm tail}$ with respect to body variables?

functional derivative of $\int A[z(t),t]dt$:

$$\frac{\delta A}{\delta z} \equiv \frac{\partial A}{\partial z} - \frac{d}{dt}\frac{\partial A}{\partial \dot{z}} + \dots$$

$$\frac{\delta H^{\mathrm{tail}}}{\delta r(t)} = -\frac{G^2 M}{5c^8}\frac{\delta}{\delta r(t)}[I_{ij}^{(3)}(t)\mathrm{Pf}_{2r(t)/c}\int_{-\infty}^{\infty}\frac{dv}{|v|}I_{ij}^{(3)}(t+v)]$$

$$\frac{\delta H^{\mathrm{tail}}}{\delta p_\varphi(t)} = -\frac{G^2 M}{5c^8}\frac{\delta}{\delta p_\varphi(t)}[I_{ij}^{(3)}(t)\mathrm{Pf}_{2r(t)/c}\int_{-\infty}^{\infty}\frac{dv}{|v|}I_{ij}^{(3)}(t+v)]$$

with, formally,

$$I_{ij}^{(3)}(t+v) = \exp\left(v\frac{d}{dt}\right)I_{ij}^{(3)}(t)$$

The Hamiltonian can also be written as

$$\begin{aligned}
H^{\mathrm{tail}} &= -\frac{G^2 M}{5c^8}I_{ij}^{(3)}(t)\mathrm{Pf}_{2r(t)/c}\int_0^{\infty}\frac{dv}{v}(I_{ij}^{(3)}(t+v)+I_{ij}^{(3)}(t-v)) \\
&= -\frac{2G^2 M}{5c^8}I_{ij}^{(3)}(t)\mathrm{Pf}_{2r(t)/c}\int_0^{\infty}\frac{dv}{v}\cosh\left(v\frac{d}{dt}\right)I_{ij}^{(3)}(t)
\end{aligned}$$

For circular orbits, $I_{ij}^{(3)}(t)$ is an eigenfunction of $\cosh\left(v\frac{d}{dt}\right)$, reading

$$\cosh\left(v\frac{d}{dt}\right) I_{ij}^{(3)}(t) = \cos\left(2v\Omega(t)\right) I_{ij}^{(3)}(t)$$

with $\Omega(t) = \frac{p_\varphi(t)}{M\nu r^2(t)} = \dot{\varphi} = \frac{\partial H_0}{\partial p_\varphi}$, $H_0 = \frac{1}{2M\nu}\frac{p_\varphi^2}{r^2} - \frac{GM\mu}{r}$, thus

$$H^{\text{tail}} = -\frac{2G^2M}{5c^8} I_{ij}^{(3)}(t) I_{ij}^{(3)}(t) \text{Pf}_{2r(t)/c} \int_0^\infty \frac{dv}{v} \cos\left(2v\frac{p_\phi(t)}{M\nu r^2(t)}\right)$$

Another writing of $\hat{H}^{\text{tail}}$ is

$$H^{\text{tail}} = -\frac{2G^2M}{5c^8} I_{ij}^{(3)}(t) \text{Pf}_{2r(t)/c} \int_0^\infty \frac{dv}{v} \cosh\left(vX(H_0)\right) I_{ij}^{(3)}(t)$$

with

$$X(H_0) = \sum_i \left(\frac{\partial H_0}{\partial p_i(t)} \frac{\partial}{\partial x^i(t)} - \frac{\partial H_0}{\partial x^i(t)} \frac{\partial}{\partial p_i(t)} \right)$$

and

$$H_0 = \frac{\mathbf{p}^2}{2\mu} - \frac{GM\mu}{r}$$

$H^{\rm tail}$ is formally local in time t, using

$$I^{(3)}_{ij}(t) = \frac{2GM}{r^2}(-4n^{<i}p^{j>} + 3(\mathbf{n}\cdot\mathbf{p})n^{<i}n^{j>})$$

For elliptical orbits it is convenient to go over to Delaunay variables, putting,

$$I_{ij}(t;e) = \sum_{p=-\infty}^{\infty} I_{ij}(p)\exp(ip\Omega t)$$

It is also more convenient to go back to the tail-Hamiltonian with $r(t)$ replaced by constant s,

$$H^{\text{tail}} = -\frac{2G^2M}{5c^8} I_{ij}^{(3)}(t)\mathrm{Pf}_{2s/c}\int_0^{\infty}\frac{dv}{v}\cosh\left(v\frac{d}{dt}\right)I_{ij}^{(3)}(t)$$

The tail action S^{tail} reads

$$S^{\text{tail}} = -\int_{t_1}^{t_2} H^{\text{tail}}dt$$

Taking $t_2 - t_1 = \mathcal{N} 2\pi/\Omega$ with $\mathcal{N}$ positive large integer (the larger $\mathcal{N}$, the less relevant the time unit $2\pi/\Omega$), the action reads

$$S^{\rm tail} = (t_2 - t_1) \frac{4G^2 M \Omega^6}{5c^8} \sum_{p=1}^{\infty} p^6 |I_{ij}(p)|^2 \mathrm{Pf}_{2s/c} \int_0^{\infty} \frac{dv}{v} \cos\left(vp\Omega\right)$$

Hereof the conserved Hamiltonian follows,

$$H^{\rm tail} = -\frac{4G^2 M \Omega^6}{5c^8} \sum_{p=1}^{\infty} p^6 |I_{ij}(p)|^2 \mathrm{Pf}_{2s/c} \int_0^{\infty} \frac{dv}{v} \cos\left(vp\Omega\right)$$