Viscosity in General Relativity

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Mathematically, the theory is very rich from both its analytic and geometric points of view. Over the past few decades, the subject of mathematical general relativity has matured into an active and exciting field of research among mathematicians.
Important features of the Minkowski metric

In special relativity, fields live in Minkowski space, which is $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ endowed with the Minkowski metric

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Causality: The four-velocity of any physical entity satisfies $|v|^2 = \eta_{\alpha\beta} v^\alpha v^\beta \leq 0$. “Nothing propagates faster than the speed of light.”
In general relativity the metric $\eta$ is no longer fixed but changes due to the presence of matter/energy: $\eta_{\alpha\beta} \rightarrow g_{\alpha\beta}(x)$, where $x = (x^0, x^1, x^2, x^3)$ are space-time coordinates.
From special to general relativity

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The values of $g_{\alpha\beta}(x)$ depend on the matter/energy near $x$. Thus distances and lengths vary according to the distribution of matter and energy on space-time. This distribution, in turn, depends on the geometry of the space-time, i.e., it depends on $g_{\alpha\beta}(x)$. 

The corresponding dynamics is governed by Einstein's equations $R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta}$. $R_{\alpha\beta}$ and $R$ are, respectively, the Ricci and scalar curvature of $g_{\alpha\beta}$, $\Lambda$ is a constant (cosmological constant), and $T_{\alpha\beta}$ is the stress-energy tensor of the matter fields.

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The Ricci and scalar curvature

The Ricci curvature of $g$ is

$$R_{\alpha\beta} = g^{\mu\nu} \left( \frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 g_{\alpha\nu}}{\partial x^\mu \partial x^\beta} - \frac{\partial^2 g_{\mu\beta}}{\partial x^\alpha \partial x^\nu} \right) + F_{\alpha\beta}(g, \partial g).$$
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Thus, Einstein’s equations are a system of second order partial differential equations for $g_{\alpha\beta}$ (and whatever other fields come from $T_{\alpha\beta}$).
Coupling gravity and matter

Consider Einstein’s equations

\[
(\ast) \begin{cases}
R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta}, \\
\nabla^\alpha T_{\alpha\beta} = 0,
\end{cases}
\]

where $\nabla$ is the covariant derivative of $g$.

Matter fields = everything that is not gravity.
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To couple Einstein’s equations to any matter field, all we need is \( T_{\alpha\beta} \).
Perfect fluid

Consider gravity coupled to a fluid: stars, cosmology.

For perfect fluids = no viscosity/no dissipation, we have the Einstein-Euler system

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where \( T_{\alpha\beta} = (p + \rho) u_\alpha u_\beta + p g_{\alpha\beta} \).

Here, \( u \) is a (time-like) unit (i.e., \( |u|^2 = g_{\alpha\beta} u_\alpha u_\beta = -1 \)) vector field representing the four-velocity of the fluid particles; \( p \) and \( \rho \) are real valued functions describing the pressure and energy density of the fluid.

The system is closed by an equation of state:

\[ p = p(\rho). \]
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Causality in general relativity

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Note that the causal structure is far more complicated than in Minkowski space since $g_{\alpha \beta} = g_{\alpha \beta}(x)$. 

Causality in GR.

A theory is causal if for any field $\phi$ its value at $x$ depends only on the "past domain of dependence of $x$."
Causality in general relativity

Causality in general relativity is formulated in the same terms as in special relativity: the four-velocity $v$ of any physical entity satisfies

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Note that that the causal structure is far more complicated than in Minkowski space since $g_{\alpha\beta} = g_{\alpha\beta}(x)$. One can better formulate causality in terms of the domain of dependence of solutions to Einstein’s equations:

A theory is causal if for any field $\varphi$ its value at $x$ depends only on the “past domain of dependence of $x$.”
What about fluids with viscosity?

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The introduction of fluids with viscosity in general relativity is well-motivated from a physical perspective:

- Real fluids have viscosity.
- Cosmology. Perfect fluids exhibit no dissipation. Maartens (’95): “The conventional theory of the evolution of the universe includes a number of dissipative processes, as it must if the current large value of the entropy per baryon is to be accounted for. (...) important to develop a robust model of dissipative cosmological processes in general, so that one can analyze the overall dynamics of dissipation without getting lost in the details of particular complex processes.”
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- The treatment of viscous fluids in the context of special relativity is also of interest in heavy-ion collisions (Rezzolla and Zanotti, '13).
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Einstein-Navier-Stokes

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All we need then is \( T^{NS}_{\alpha\beta} \) (\( T_{\alpha\beta} \) for Navier-Stokes).
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Critical points of an action $S$ give equations of motion. For example:

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The action $S$ also determines $T_{\alpha\beta}$. 
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The stress-energy tensor is given by

$$T_{\alpha\beta} = \frac{1}{\sqrt{-\det(g)}} \frac{\delta \mathcal{L}}{\delta g^{\alpha\beta}}.$$
We have seen that in order to couple Einstein’s equations to the Navier-Stokes equations all we need is $T^{NS}_{\alpha\beta}$. Problem: the Navier-Stokes equations do not come from an action principle. Therefore, we do not know what $T^{NS}_{\alpha\beta}$ is, or how to couple it to Einstein’s equations.

Remark: stress-energy for the Navier-Stokes equations in non-relativistic physics is constructed “by hand.”
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Ad hoc construction

We can still postulate a $T_{\alpha\beta}^{NS}$ and couple it to Einstein’s equations.
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Eckart ('40) proposed the following stress-energy tensor for a relativistic viscous fluid

$$T^E_{\alpha\beta} = (p + \varrho) u_\alpha u_\beta + pg_{\alpha\beta} - (\zeta - \frac{2}{3} \vartheta) \pi_{\alpha\beta} \nabla_\mu u^\mu$$

$$- \vartheta \pi^\mu_\alpha \pi^\nu_\beta (\nabla_\mu u_\nu + \nabla_\nu u_\mu) - \kappa (q_\alpha u_\beta + q_\beta u_\alpha),$$

where $\pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$, $\zeta$ and $\vartheta$ are the coefficients of bulk and shear viscosity, respectively, $\kappa$ is the coefficient of heat conduction, and $q_\alpha$ is the heat flux.
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$$- \vartheta \pi_{\alpha\beta}^{\mu\nu} (\nabla_\mu u_\nu + \nabla_\nu u_\mu) - \kappa (q_\alpha u_\beta + q_\beta u_\alpha),$$

where $\pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$, $\zeta$ and $\vartheta$ are the coefficients of bulk and shear viscosity, respectively, $\kappa$ is the coefficient of heat conduction, and $q_\alpha$ is the heat flux.

$T_{\alpha\beta}^E$ reduces to the stress-energy tensor for a perfect fluid when $\zeta = \vartheta = \kappa = 0$, it is a covariant generalization of the non-relativistic stress-energy tensor for Navier-Stokes, and satisfies basic thermodynamic properties.
Lack of causality

Hiscock and Lindblom ('85) have shown that a large number of choices of viscous $T_{\alpha\beta}$, including Eckart’s proposal, leads to theories that are not causal and unstable.

Two possible choices to circumvent this problem are:

1. Extend the space of variables of the theory, introducing new variables and equations based on some physical principle. Second order theories.

2. Find a stress-energy tensor that avoids the assumptions of Hiscock and Lindblom. First order theories.

Despite the results of Hiscock and Lindblom, $T_{\alpha\beta}$ is still used in applications (particularly in cosmology) for the construction of phenomenological models.
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Despite the results of Hiscock and Lindblom, $T_{\alpha\beta}^E$ is still used in applications (particularly in cosmology) for the construction of phenomenological models.
Define the **entropy current** as

\[ S^\alpha = s n u^\alpha + \kappa \frac{q^\alpha}{T}, \]

where \( s \) is the specific entropy, \( n \) is the rest mass density, and \( T \) is the temperature.
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Equation (1) cannot be assumed. Rather, it has to be verified as a consequence of the equations of motion. This is one of the main constraints for the construction of relativistic theories of viscosity.
A widely studied case of second order theories is the Mueller-Israel-Stewart (MIS) ('67, '76, '77).

Consider a stress-energy tensor of the form

\[ \tilde{T}_{\alpha\beta} = (p + \rho) u_{\alpha} u_{\beta} + \pi_{\alpha\beta} \Pi + \pi_{\alpha\beta} \Pi + Q_{\alpha} u_{\beta} + Q_{\beta} u_{\alpha} . \]

\( \Pi, \Pi_{\alpha\beta}, \) and \( Q_{\alpha} \) correspond to the dissipative contributions to the stress-energy tensor.

Setting \( \Pi = -\zeta \nabla_{\mu} u_{\mu}, Q_{\alpha} = -\kappa q_{\alpha}, \) and \( \Pi_{\alpha\beta} = -\vartheta \pi_{\mu \alpha} \pi_{\nu \beta} (\nabla_{\mu} u_{\nu} + \nabla_{\nu} u_{\mu} - \frac{2}{3} \nabla_{\mu} u_{\mu}) \) gives back \( T_{E\alpha\beta} \).

In the MIS theory, the quantities \( \Pi, \Pi_{\alpha\beta}, \) and \( Q_{\alpha} \) are treated as new variables on the same footing as \( \rho, u_{\alpha}, \) etc.
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The new variables $\Pi$, $\Pi_{\alpha\beta}$, and $Q_\alpha$ require the introduction of further equations of motion.

In the MIS theory, one postulates an entropy current of the form

$$ S_\alpha = s_{nu\alpha} + Q_\alpha T - \left( \beta_0 \Pi^2 + \beta_1 Q_\mu Q^\mu + \beta_2 \Pi_{\mu\nu} \Pi^{\mu\nu} \right) u_\alpha T + \alpha_0 \Pi Q_\alpha T + \alpha_1 \Pi_{\alpha\mu} Q^\mu, $$

for some coefficients $\beta_0$, $\beta_1$, $\beta_2$, $\alpha_0$, and $\alpha_1$.

Next, we compute $\nabla_\alpha S_\alpha$ and seek the simplest relation, linear in the variables $\Pi$, $\Pi_{\alpha\beta}$, and $Q_\alpha$, which assures that the second law of thermodynamics $\nabla_\alpha S_\alpha \geq 0$ is satisfied. This gives equations for $\Pi$, $\Pi_{\alpha\beta}$, and $Q_\alpha$ that are appended to Einstein’s equations.
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Summary of results for second order theories

For the MIS and other second order theories:

▶ Causality for certain values of the variables.

On the other hand:

▶ The physical content of the $\alpha_i$ and $\beta_i$ coefficients in is not apparent (although it can be in some cases).

▶ The equations for $\Pi$, $\Pi_{\alpha\beta}$, and $Q_\alpha$ are ultimately arbitrary.

▶ Non-relativistic limit?

▶ No "strong shock-waves solutions."

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Lichnerowicz proposed the following stress-energy tensor for a relativistic viscous fluid:

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T_{\alpha\beta} = (p + \varrho)u_\alpha u_\beta + pg_{\alpha\beta} - \left(\zeta - \frac{2}{3}\vartheta\right)\pi_{\alpha\beta} \nabla_\mu C^\mu \\
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Lichnerowicz’s stress-energy tensor had been mostly ignored for many years, but recently it has been showed as potentially viable candidate for relativistic viscosity.
Some results

Using Lichnerowicz’s stress-energy tensor, it is possible to show (D–, ’14; D– and Czubak ’16; D–, Kephart, and Scherrer, ’15):

- The equations of motion are causal, including when coupling to Einstein’s equations.
- This holds under the assumption that the fluid is irrotational or under restrictions on the initial data (+ other hypotheses).
- For certain values of the variables, the second law of thermodynamics is satisfied.
- The correct non-relativistic limit is obtained.
- Existence of solutions (including coupling to Einstein’s equations).
- Applications to cosmology lead to different models, in particular big-rip scenarios.

None of these results consider all dissipative variables (e.g. shear viscosity but no bulk viscosity, etc).
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Geroch and Lindblom (’90) developed a general framework for theories of relativistic viscosity that leads to causal dynamics under many circumstances. One then has to show that a particular theory (e.g. MIS) fits in the formalism under the conditions that give rise to causality.
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Possible route: promote Lichnerowicz’s approach to a second order theories.

Can numerical works help?
Are we missing some fundamental insight? What are the correct guiding principles to approach the problem? E.g., should we enforce the second law to all orders? Should we develop causality instead and try to prove the second law a posteriori? A mix of both? Or yet should we give up the hope for a general theory and make a case-by-case analysis?

Possible route: promote Lichnerowicz’s approach to a second order theories.

Can numerical works help?

– Thank you for your attention –